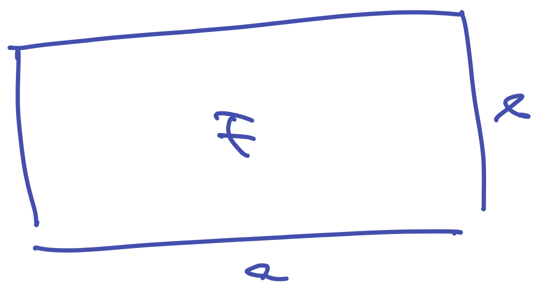
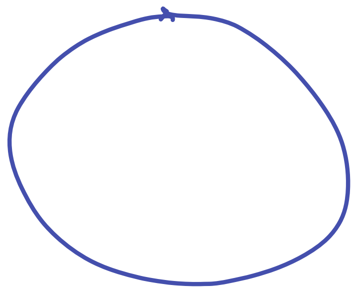
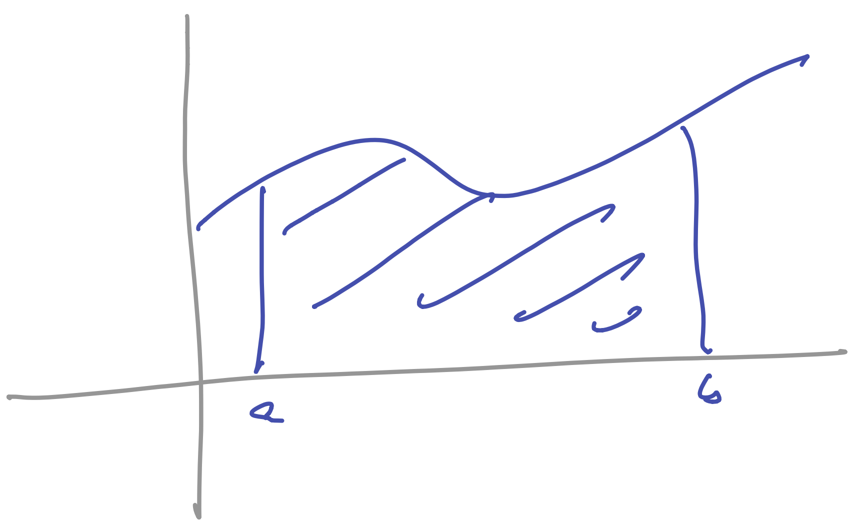
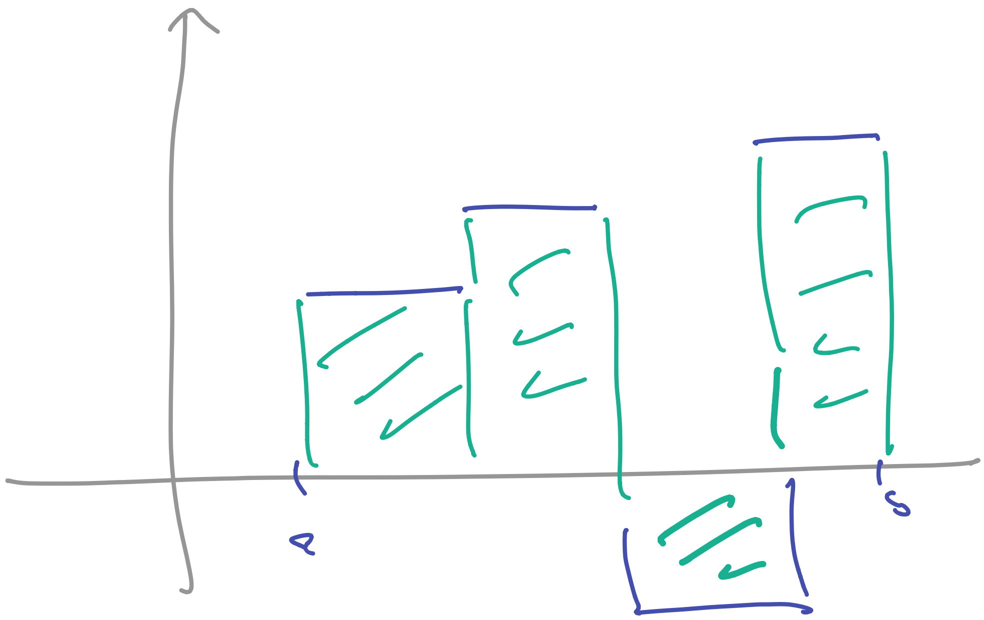
1. Vorlesung 19. April 2021



$F = ab$

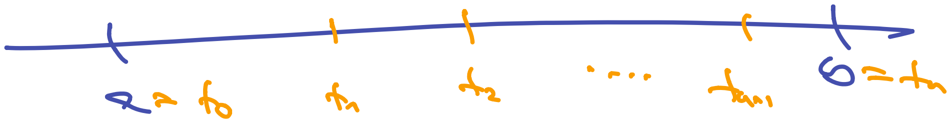






Treppenfunktion

$[a,b] \subset \mathbb{R}$



$Z = (t_0, t_1, \dots, t_n)$:

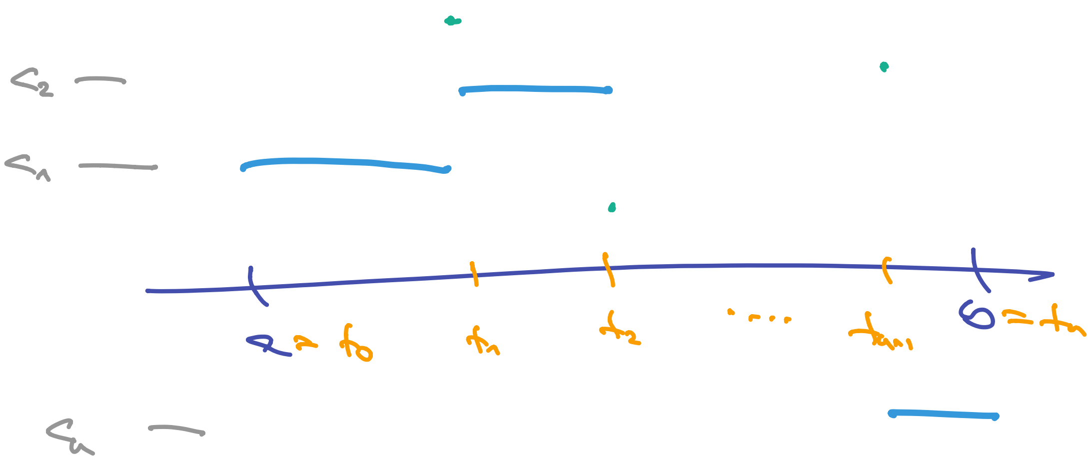
$a = t_0 < t_1 < t_2 < \dots < t_n = b$

$\varphi : [a,b] \longrightarrow \mathbb{R}$ heißt Treppenf.

wenn es eine Zerlegung $(t_0, \dots, t_n)$ von $[a,b]$ gibt, so dass

$$\varphi\big|_{(t_{k-1}, t_k)} = c_k, \quad k = 1, \dots, n.$$

Raum: $T_a^b = T([a,b])$.



$$\boxed{\varphi = \sum_{k=1}^{n} c_k \chi_{(t_{k-1}, t_k)}}$$

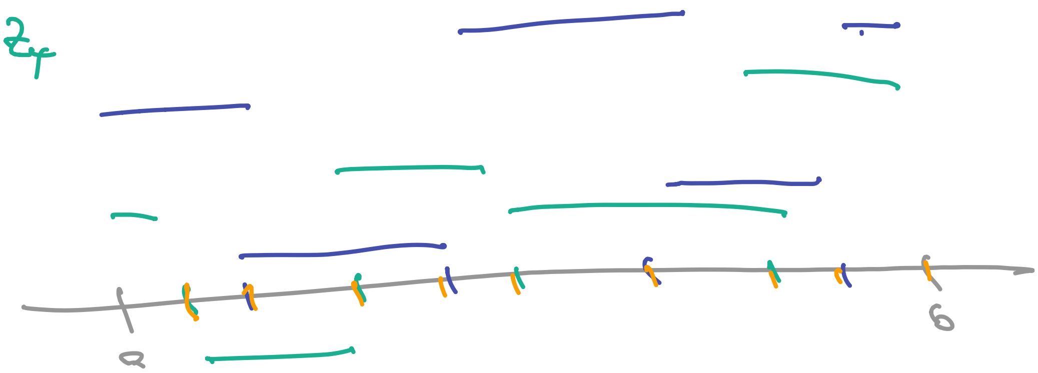
Bem.: Jede Tf. ist glm. beschränkt:

$$\|\varphi\|_{[a,b]} = \sup_{t\in[a,b]} |\varphi(t)| < \infty$$

Also: $T_a^b \subset B_a^b = B([a,b])$.

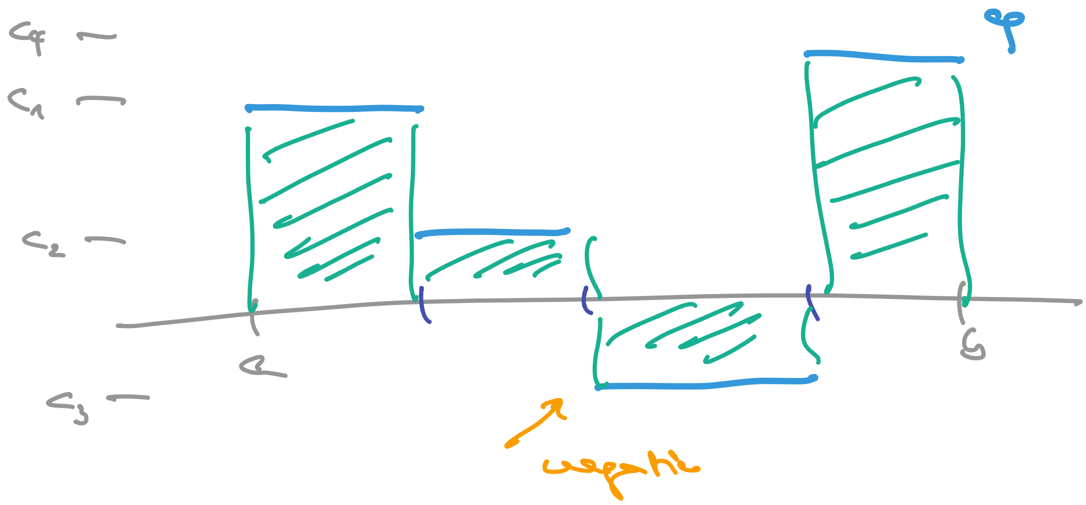
$\varphi, \psi \in T_a^b$

Z_φ Z_ψ



$Z = Z_\varphi \cup Z_\psi$ Verfeinerung von Z_φ und Z_ψ.

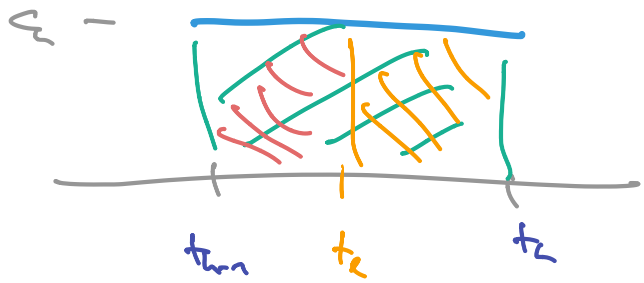
$$\int_a^b (\varphi) := \sum_{k=1}^{n} c_k (t_k - t_{k-1})$$



Begründung! Seien φ_1, φ_2 zwei Treppenfunktionen, wie die Partition identisch war.

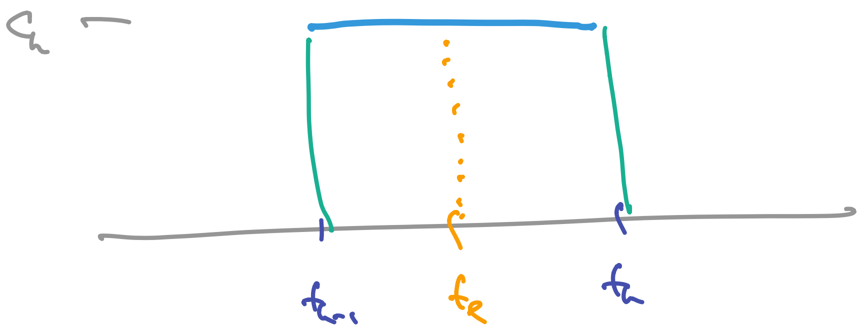
Z_1 und Z_2 seien ihre Zerlegungen.

Dann $Z = Z_1 \cup Z_2$



$$c_k \cdot (t_k - t_{k-1})$$

$$= \underline{c_k \cdot (t_k - t_\ell)} + \underline{c_k \cdot (t_\ell - t_{k-1})}$$



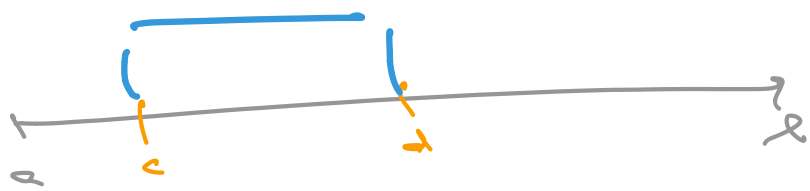
$$\int_a^b : T_a^b \longrightarrow \mathbb{R}$$

$$\varphi \longmapsto \int_a^b(\varphi)$$

Funktional.

Beispiele:

1. $I \subset (a,b)$



$I = [c,d]$,

$$\chi_I = \begin{cases} 1 & \text{auf } (c,d) \\ 0 & \text{auf } (a,b)\setminus(c,d) \end{cases}$$

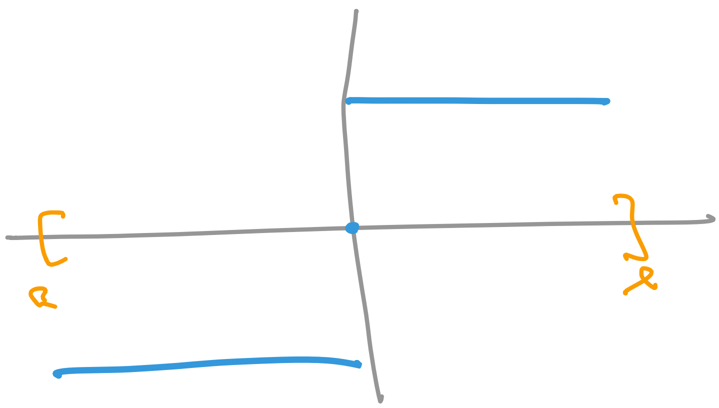
Treppenfunktion:

$$\int_a^b(\chi_I) = 1\cdot(d-c) = d-c$$

$$= |I|.$$

Länge von I.

2. $\operatorname{sgn} = \begin{cases} 1 & \text{auf } (0,\infty) \\ 0 & \text{auf } \{0\} \\ -1 & \text{auf } (-\infty,0) \end{cases}$



Gauss $[\cdot]$

3. Jede Funktion φ_0, die an alle außer endlich viele Punkte in $[a,b]$ gleich $= 0$ ist, ist eine TF, und $\int_a^b (\varphi) = 0$.

Satz: $\int_a^b : T_a^b \longrightarrow \mathbb{R}$

(i) **Linearität:**

$$\int_a^b(\lambda\varphi + \mu\psi) = \lambda \int_a^b(\varphi) + \mu \int_a^b(\psi).$$

(ii) **Monotonie:** $\varphi \le \psi$

$$\Longrightarrow \int_a^b(\varphi) \le \int_a^b(\psi).$$

(iii) **Normierung:**

$$\varphi = \chi_{[a,b]} \Longrightarrow \int_a^b(\varphi) = b-a.$$

(iv) **Lipschitzstetigkeit:**

$$\left| \int_a^b(\varphi) - \int_a^b(\psi) \right| \le \underbrace{(b-a)}_{\text{Lipschitz}} \cdot \underbrace{\|\varphi-\psi\|_{[a,b]}}_{\text{"Abstand von } \varphi \text{ zu } \psi\text{"}}$$

Bew.:

$$\left|\int_a^b(\varphi)\right| = \left|\sum_{k=1}^n c_k (t_k - t_{k-1})\right|$$

$$\leq \sum_{k=1}^n |c_k| \underbrace{(t_k - t_{k-1})}_{\geq 0\ !}$$

$$\leq \max\{|c_1|, \dots, |c_n|\} \cdot \underbrace{\sum_{k=1}^n (t_k - t_{k-1})}$$

$$= \|\varphi\|_{[a,b]} \cdot (b-a)$$

Wegen Linearität:

$$\left|\int_a^b(\varphi) - \int_a^b(\psi)\right|$$

$$= \left|\int_a^b(\varphi - \psi)\right|$$

$$\underset{\text{s.o.}}{\leq} (b-a) \cdot \|\varphi - \psi\|_{[a,b]}. \qquad \square$$

$f \in \mathcal{R}_a^b$ Regelfunktion,

Gibt es (φ_n) in $\mathcal{T}_a^b$ mit

$$\varphi_n \rightrightarrows f$$

glm. Konvergenz: $\|f - \varphi_n\|_{[a,b]} \to 0$.

Def.:

$$J_a^b(f) := \lim_{n\to\infty} J_a^b(\varphi_n)$$

Wohldefiniertheit: Angenommen:

$$\varphi_n \rightrightarrows f, \quad \psi_n \rightrightarrows f.$$

Dann auch

$$\|\varphi_n - \psi_n\|_{[a,b]} \longrightarrow 0$$

$$\left| J_a^b(\varphi_n) - J_a^b(\psi_n) \right|$$

$$= \left| J_a^b(\varphi_n - \psi_n) \right|$$

$$\leq (b-a) \cdot \|\varphi_n - \psi_n\|_{[a,b]}$$

$$\longrightarrow 0$$

Also: $\lim J_a^b(\varphi_n) = \lim J_a^b(\psi_n)$. □

Beweis von Rechengesetze:

- Linearität: $f, g \in R_a^b$:

$$\varphi_n \rightrightarrows f$$

$$\psi_n \rightrightarrows g$$

dann : $\alpha\varphi_n + \beta\psi_n \rightrightarrows \alpha f + \beta g$:

$$\int_a^b (\alpha f + \beta g) = \lim \int_a^b (\alpha\varphi_n + \beta\psi_n)$$

$$= \lim \left(\alpha \int_a^b (\varphi_n) + \beta \int_a^b (\psi_n)\right)$$

$$= \alpha \cdot \lim \int_a^b (\varphi_n) + \beta \cdot \lim (\;)$$

$$= \alpha \cdot \int_a^b (f) + \beta \cdot \int_a^b (g).$$

- Monotonie: Wegen Linearität genügt:

z.z.: $f \in R_c^\infty$ und $f \geq 0$

$\Rightarrow \int_c^\infty (f) \geq 0.$

Nun, gilt $\varphi_n \rightrightarrows f$,

so gilt für $f \geq 0$:

$\varphi_n^+ \rightrightarrows f$,

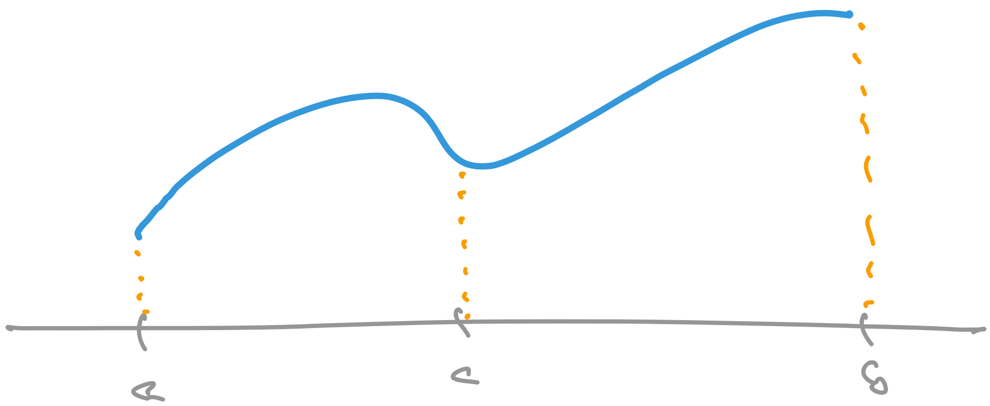
wobei $\underbrace{\varphi_n^+ = \max(\varphi_n, 0)}_{\text{Treppenfunktion}} \geq 0$.

Wegen: $\int_c^\infty(\varphi_n^+) \geq 0$

$\Rightarrow \lim \int_c^\infty(\varphi_n^+) = \int_c^\infty(f) \geq 0.$

- Normierung: Siehe oben.

- Lipschitz: Ü.



" $\int_a^b = \int_a^c + \int_c^b$ "

Beweis: Sei Treppenf. $\varphi \in T_a^b$:

$$\Rightarrow \quad \varphi|_{[a,c]} \in T_a^c \,, \qquad \varphi|_{[c,b]} \in T_c^b$$

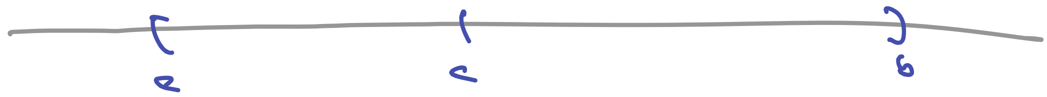
und:

$$\int_a^b(\varphi) = \int_a^c(\varphi) + \int_c^b(\varphi).$$

Damit auch:

$$\begin{aligned}
\int_a^b(f) &= \lim \int_a^b(\varphi_n) \\
&= \lim \left(\int_a^c(\varphi_n) + \int_c^b(\varphi_n) \right) \\
&= \lim \int_a^c(\varphi_n) + \lim \int_c^b(\varphi_n) \\
&= \int_a^c(f) + \int_c^b(f) \,. \quad \square
\end{aligned}$$

$a < c < b$:



Konvention: $a = b$: $\int_a^a (f) = 0$

$b < a$: $\int_b^a (f) = - \int_a^b (f)$.

Dann gilt immer:

$$\boxed{\int_a^b (f) = \int_a^c (f) + \int_c^b (f).}$$

Bsp: $c < a < b$:

$$\int_c^b (f) = \int_c^a (f) + \int_a^b (f)$$

$$\Rightarrow \int_a^b (f) = \int_c^b (f) - \int_c^a (f)$$

$$= \int_c^b (f) + \int_a^c (f)$$

$$= \int_a^c (f) + \int_c^b (f)$$

$$\boxed{\int_a^b f = \int_a^b (f) .}$$

$$\int_a^b f = \int_a^c f + \int_c^b f$$

$$\left|\int_a^b f\right| \le \int_a^b |f| \quad :$$

Beweis: $f \in R_a^b$, also

$\varphi_n \rightrightarrows f$

Dann:

$$\underset{\in T_a^b}{|\varphi_n|} \rightrightarrows \underset{\in R_a^b}{|f|} \qquad Ü$$

Weiter:

$$\left|\int_a^b \varphi_n\right| = \left|\sum_{k=1}^n c_k (t_k - t_{k-1})\right|$$

$$\le \sum_{k=1}^n |c_k| (t_k - t_{k-1})$$

$$= \int_a^b |\varphi_n|$$

Limes $n \to \infty$:

$$\left|\int_a^b f\right| \le \int_a^b |f|. \qquad \square$$