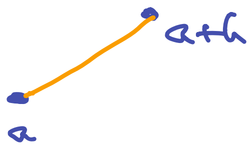
15. Vorlesung 14. 6. 2021



$$\varphi(t) = f(a+th), \quad 0 \le t \le 1$$

φ bei 0 in Taylorreihe entwickeln und bei 1 auswerten.

Beweis:

$$\varphi : \mathbb{R} \to W$$

$$\varphi(t) := f(a+th) ,$$

Dann Kettenregel:

$$\varphi'(t) = Df(a+th) \cdot h$$

$$= \partial_h f(a+th)$$

Mit Induktion!

$$\varphi^{(k)}(t) = \partial_h^k f(a+th) . \quad \Big|_{t=0}$$

Mit Taylor:

$$\varphi(x) = \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} \cancel{x^k} +$$

$$\frac{1}{r!} \int_0^x (x-t)^r \varphi^{(r+1)}(t) \, dt$$

Für $n=1$:

$$\varphi'(t) = Df(a+th) h$$

$$= f'(a+th) \cdot h$$

$$\vdots$$

$$\varphi^{(k)}(t) = f^{(k)}(a+th) \, h^k$$

Also:

$$\partial_h^k f(a) = f^{(k)}(a) \cdot h^k .$$

Multiindex:

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n .$$

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \prod_{i=1}^{n} x_i^{\alpha_i}$$

Vereinbarung: $x_i^0 = 1$.

$$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} = \prod_{i=1}^{n} \partial_i^{\alpha_i}$$

Vereinbarung: $\partial_i^0 = I$.

$$\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f = \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f$$

$$\alpha! := \alpha_1! \cdots \alpha_n! = \prod_{i=1}^{n} \alpha_i!$$

$$|\alpha| := \alpha_1 + \dots + \alpha_n$$

≥ 0

Bsp.: $f \in C^4(\mathbb{R}^3)$, $\alpha = (3,1,0)$:

$$\frac{1}{\alpha!}\partial^\alpha f = \frac{1}{3!\,1!\,0!} f_{xxxy}$$

$$= \frac{1}{6} f_{xxxy}$$

$\beta = (1,0,2)$:

$$\frac{1}{\beta!}\partial^\beta f = \frac{1}{2} f_{xzz}.$$

Beweis:

$$(\lambda_1 + \dots + \lambda_n)^{m} = \underbrace{(\lambda_1 + \dots + \lambda_n) \cdot \ldots \cdot (\lambda_1 + \dots + \lambda_n)}_{m\text{-mal}}$$

ausmultiplizieren

$$= \sum_{i_1, \dots, i_m = 1}^{n} \lambda_{i_1} \cdots \lambda_{i_m}$$

Gleiche Summand: $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$

$$\frac{m!}{\alpha_1! \cdots \alpha_n!}$$

$$= \sum_{|\alpha| = m} \frac{m!}{\alpha_1! \cdots \alpha_n!} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$$

$$= \sum_{|\alpha| = m} \frac{m!}{\alpha!} \lambda^{\alpha} \qquad \blacksquare$$

Klassische Bin. Formel : $n = 2$

$$\alpha = (\alpha_1, \alpha_2)$$

$$|\alpha| = \alpha_1 + \alpha_2 = m$$

Also:

$$\alpha_1 = k \;, \quad \alpha_2 = m - k \;:$$

$$(x_1 + x_2)^m = \sum_{k=0}^{m} \underbrace{\frac{m!}{k!\,(m-k)!}}_{\binom{m}{k}} x_1^k \, x_2^{m-k}$$

Dem.:

$m=0$: (caso limite) ✓

$m=1$:

$$\frac{1}{m!}\partial_h^m = \partial_h$$

$$= h_1\partial_1 + \dots + h_n\partial_n$$

$$= \sum_{|\alpha|=1} h^\alpha \partial^\alpha .$$

$\alpha = (0, \dots, 1, \dots)$

$m \geq 2$:

$$\frac{1}{m!}\partial_h^m = \frac{1}{m!}(h_1\partial_1 + \dots + h_n\partial_n)^m$$

commutano

$$= \sum_{|\alpha|=m} \frac{1}{\alpha!} \underbrace{(h_1\partial_1)^{\alpha_1} \cdots (h_n\partial_n)^{\alpha_n}}_{h^\alpha \partial^\alpha}$$

$$= \sum_{|\alpha|=m} \frac{1}{\alpha!} h^\alpha \partial^\alpha . \qquad \blacksquare$$

$$\frac{1}{k!}\,\partial_h^k f(a) = \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!}\,\partial^\alpha f(a)$$

$$= \sum_{|\alpha|=k} \frac{1}{\alpha!}\,\partial^\alpha f(a)\cdot h^\alpha$$

$$\sum_{k=0}^{r} \underline{\qquad}\cdot\underline{\qquad} = \sum_{|\alpha|\le r} \underline{\qquad\quad}\cdot\underline{\qquad\quad}$$

$$\boxed{\frac{1}{k!}\underbrace{\partial_h^k f(a)} = \sum_{|\alpha|=k} \frac{1}{\alpha!}\partial^\alpha f(a)\cdot h^\alpha}$$

$$R_a^r f(h) = \frac{1}{\cancel{r!}}\int_0^1 (1-t)^r\,(r+1)\cancel{!}\sum_{|\alpha|=r+1}\frac{h^\alpha}{\alpha!}\partial^\alpha f(a+th)\,dt$$

$$= (r+1)\cdot\int_0^1 \underbrace{(1-t)^r}_{\geq 0}\ \underbrace{\sum_{|\alpha|=r+1}\frac{h^\alpha}{\alpha!}\partial^\alpha f(a+th)}_{\text{Skalar, stetig}}\,dt$$

$$= (r+1)\cdot\sum_{|\alpha|=r+1}\frac{h^\alpha}{\alpha!}\int_0^1 (1-t)^r\,\partial^\alpha f(a+th)\,dt$$

Skalar:

$\theta\in[0,1]$:

$$= \cancel{(r+1)}\cdot\sum_{|\alpha|=r+1}\frac{h^\alpha}{\alpha!}\partial^\alpha f(a+\theta h)\cdot\cancel{\underbrace{\int_0^1 (1-t)^r\,dt}_{\frac{1}{r+1}}}$$

$$= \sum_{|\alpha|=r+1}\frac{h^\alpha}{\alpha!}\partial^\alpha f(\xi),$$

$$\xi\in[a,a+h]$$

Bsp:

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$f(x) = \frac{1}{2} \langle Ax, x \rangle = \frac{1}{2} \sum_{k,l=1}^{n} a_{kl} x_k x_l$$

mit $A = (a_{kl})$ symmetrisch.

Dann

$$f_{x_k x_l} = a_{kl} ,$$

Also: $Hf = A$.

Dem:

$$f(x+h) = \overset{\alpha=0}{f(x)} + \sum_{|\alpha|=1} \partial^\alpha f(x)\, h^\alpha + \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial^\alpha f(\xi)\, h^\alpha$$

$$= \sum_{k=1}^{n} f_{x_k}(x) \cdot h_k$$

$$= \langle \nabla f(x), h \rangle$$

$$= \frac{1}{2} \partial_h^2 f(\xi) = \frac{1}{2} \sum_{k,l=1}^{n} f_{x_k x_l}(\xi)\, h_k h_l$$

$$= \frac{1}{2} \langle Hf(\xi) h, h \rangle$$

Bemerkung: Gilt auch hier:

$$\varphi(t) = f(a+th)$$

Taylor:

$$\varphi(1) = \varphi(0) + \varphi'(0)\cdot 1 + \int_0^1 (1-t)\,\varphi''(t)\,dt$$

$$Df(a)h = \langle \nabla f(a), h \rangle$$

$$R(1) = \varphi''(\tau) \cdot \int_0^1 (1-t)\,dt$$

$$= \frac{1}{2} \cdot \langle Hf(\xi)h, h \rangle. \qquad \blacksquare$$

Bsp.:

xy^2z^3 Monom vom Grad 6

$1 + x + xy^2 + xy^2z^3$ Polynom vom Grad 6

$c \in \mathbb{R}^*$ $= \mathbb{R}\setminus\{0\}$ Konstantes Polynom vom Grad 0

$c = 0$ also $p \equiv 0$:

$$\begin{aligned}\text{Grad} &= \max\{\,|\alpha| : a_\alpha \neq 0\,\}\\ &= \max \emptyset\\ &= -\infty.\end{aligned}$$