20. Vorlesung 5. 7. 2021

Bew: Angenommen: $\varphi_1, \dots, \varphi_n$ Lösungen von $\dot{x} = Ax$,

l.a. Sei $t = t_0$:

$$\varphi(t) = \alpha_1 \varphi_1(t) + \dots + \alpha_n \varphi_n(t)$$

$$\varphi(t_0) = 0.$$

Lösung von $\dot{x} = Ax$,

$$\varphi(t_0) = 0.$$

Also: $\varphi(t) = 0, \quad t \in \mathbb{R}$ $\square$

Beweis: Jede Lösung φ von $\dot{x} = Ax$ ist eindeutig durch AW $\varphi(0)$.

Also:

$$\varphi(0) = \alpha_1 \varphi_1(0) + \dots + \alpha_n \varphi_n(0)$$

Setze:

$$\psi(t) = \alpha_1 \varphi_1(t) + \dots + \alpha_n \varphi_n(t)$$

Dann:

$\psi(t)$ Lösung

$$\psi(0) = \varphi(0)$$

$$\Rightarrow \quad \psi(t) = \varphi(t) = \dots \qquad \square$$

$$\det(AB) = \det(BA)$$

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Seien B, C zwei Darstellungsmatrizen von A.

Dann

$$C = T^{-1} B T$$

Also:

$$\operatorname{tr}(C) = \operatorname{tr}(T^{-1} B T) = \operatorname{tr}(B T T^{-1}) = \operatorname{tr}(B).$$

Proof:

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots$$
$$= I + t\left(A + \frac{1}{2}A^2 t + \dots\right)$$
$$= I + t\,A(t)$$

$$A(t) = \sum_{k \geq 1} \frac{t^{k-1}}{k!} A^k, \quad A(0) = A.$$

$$e^{At} = [I_1 + tA_1(t), \dots, I_n + tA_n(t)].$$

[illegible]:

$$\det e^{At}$$
$$= \det[I_1 + tA_1(t), \dots, I_n + tA_n(t)].$$
$$= \det(I_1, \dots, I_n)$$
$$+ \ t \sum_{i=1}^{n} \det(I_1, \dots, A_i(t), \dots, I_n)$$
$$+ \ O(t^2)$$

i.e. trace of $A(t)$

Bew.:

$$\det e^{At} = 1 + t \sum_{i=1}^{n} a_{ii}(A) + O(t^2)$$

$$= 1 + t \cdot \sum_{i=1}^{n} A_{ii} + O(t^2)$$

$$= 1 + \underbrace{t \cdot \operatorname{tr}(A)} + O(t^2)$$

Also:

$$\frac{d}{dt} \det e^{At} \Big|_{t=0} = \operatorname{tr}(A). \qquad \square$$

Beweis:

$$d(t) := \det e^{At}:$$

$$\begin{aligned} d(s+t) &= \det e^{A(s+t)} \\ &= \det e^{As+At} \\ &= \det e^{As} e^{At} \\ &= \det\left(e^{As}\right) \cdot \det\left(e^{At}\right) \\ &= d(s) \cdot d(t) \end{aligned}$$

1-Param.-Gruppe.

$$\begin{aligned} \frac{d}{dt} d(t) &= \frac{d}{ds} d(s+t) \Big|_{s=0} \\ &= \frac{d}{ds} d(s)\, d(t) \Big|_{s=0} \\ &= \frac{d}{ds} d(s) \Big|_{s=0} d(t) \\ &\overset{=}{\scriptstyle \text{Lemma}} \operatorname{Sp}(A) \cdot d(t). \end{aligned}$$

Also:

$$\dot{d}(t) = \mathrm{Sp}(A)\, d(t)$$

$$d(0) = \det \exp e^{A0} = 1$$

Also:

$$d(t) = e^{\mathrm{Sp}(A)\cdot t}.$$ $\square$

e^{At} Zeit-t-Abb zu $\dot{x} = Ax$

$$e^{At}: \quad x_0 \longmapsto e^{At} x_0$$

$\det e^{At}$ Volumenverzerrung:

$$A : U \to U$$

$$K \subset U$$

$$\mathrm{vol}(AK) = \det(A) \cdot \mathrm{vol}(K).$$

$$(Ty)' = Ty' = TBy$$

$$(Ty)' = x' = Ax = ATy$$

Also:

$$TBy = ATy, \qquad y \in U$$

Also

$$TB = AT$$

$$\boxed{B = T^{-1} A T}$$

(i) real distinct

(ii) real repeated } Diagonal / Jordan

(iii) complex conjugate

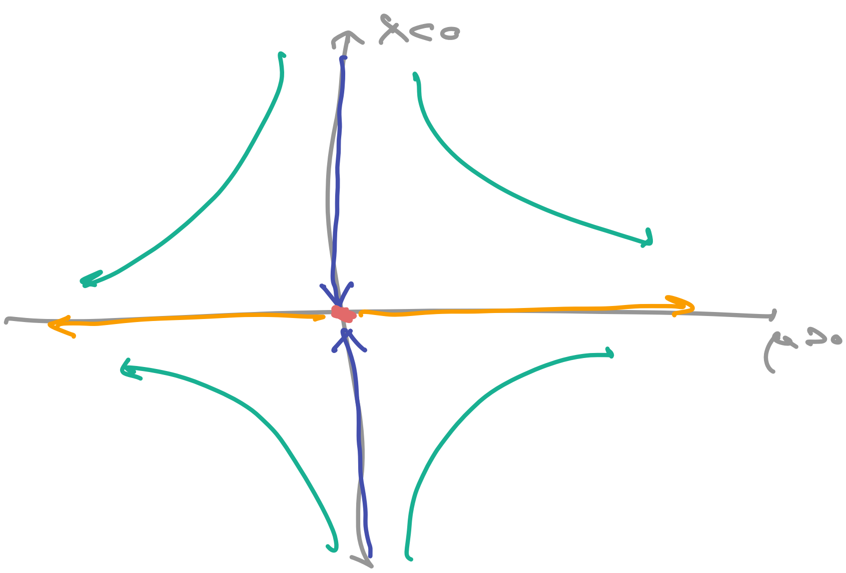
$$\varphi(t) \equiv 0$$

$$x(t) = e^{At} x_0$$

$$= \begin{pmatrix} e^{\lambda t} & \\ & e^{\mu t} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

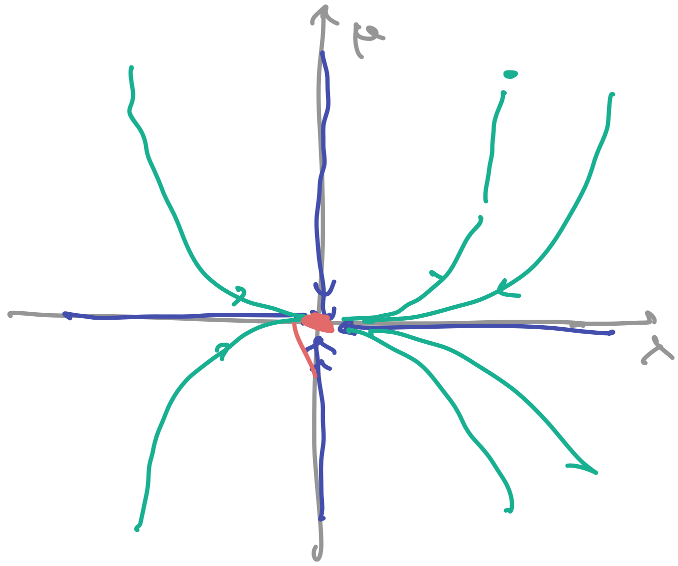
$$= \begin{pmatrix} e^{\lambda t} a \\ e^{\mu t} b \end{pmatrix}, \quad a, b \in \mathbb{R}$$

$\lambda\mu < 0$: λ, μ verschieden

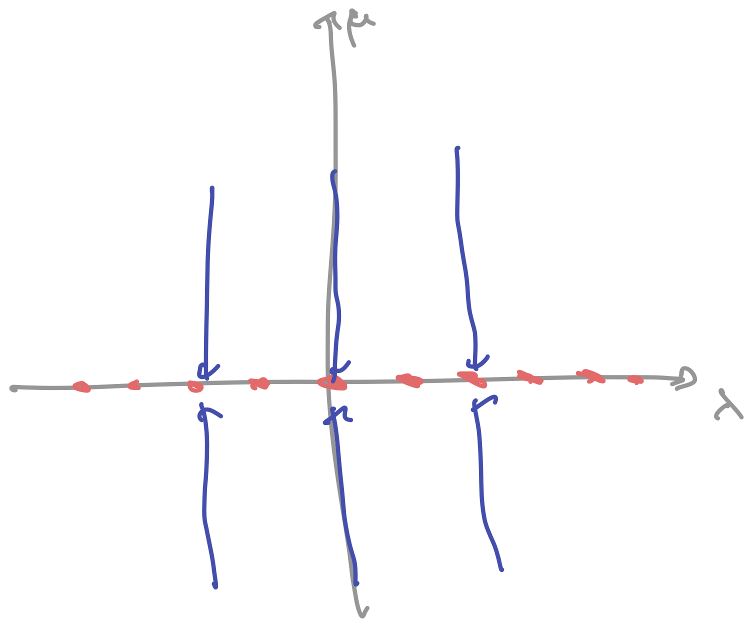


Sattel

$\mu < \lambda < 0$ stabiler Knoten



$\mu < 0 = \lambda$ entartet



Bew:

$$T^{-1} A T = \mathrm{diag}(\lambda, \mu)$$

$$= \lambda \cdot I \qquad \text{für } \lambda = \mu$$

$$\Rightarrow \quad A = T \lambda I T^{-1}$$

$$= \lambda T T^{-1}$$

$$= \lambda I$$

(ii) A diagonalisierbar:

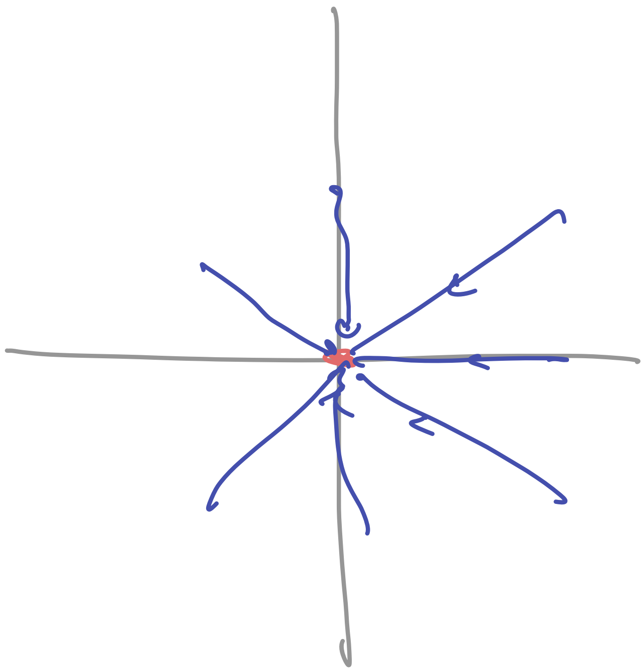
$$A = \lambda I$$

Also Lösung :

$$\varphi(t) = e^{\lambda t} x_0$$

Gerade

$x<0$:



A Jordan Block:

$$A = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$$

$$= \underbrace{\lambda \cdot I}_{} + \underbrace{N}_{}, \qquad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Also:

$$N^2 = 0$$

$$e^{At} = e^{(\lambda I + N)t}$$

$$= e^{\lambda t} \cdot e^{Nt}$$

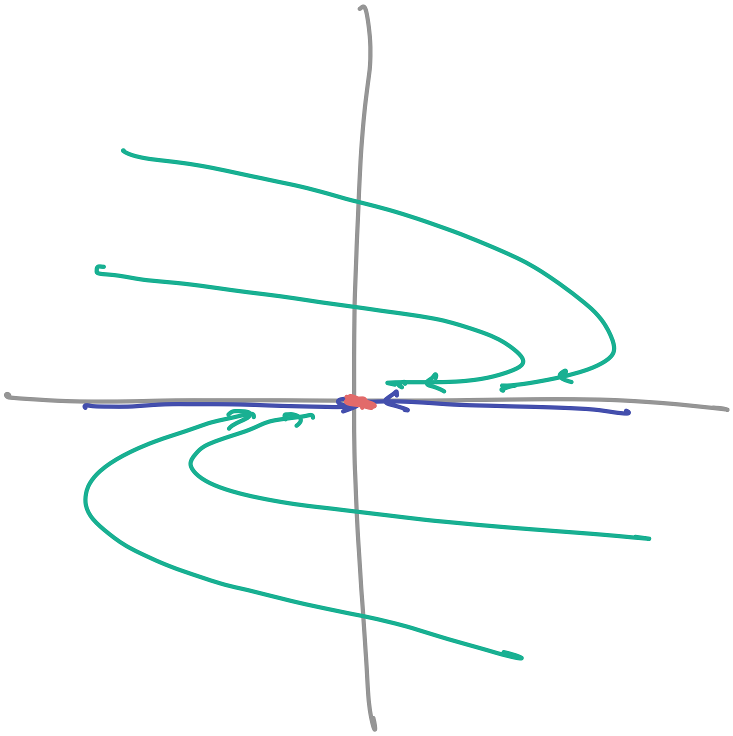
$$= e^{\lambda t}(I + Nt)$$

$$= e^{\lambda t} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$$

$$\varphi(t) = e^{\lambda t} \begin{pmatrix} a + bt \\ b \end{pmatrix} \qquad x = g(y)$$

$$\text{AW} \quad \begin{pmatrix} a \\ b \end{pmatrix}$$

$\lambda < 0$



[illegible] stabile [illegible]

Bew.: $\alpha + i\omega$ EV: $u + iv$:

$$A(u+iv) = (\alpha+i\omega)(u+iv)$$
$$= (\alpha u - \omega v) + i(\omega u + \alpha v)$$

Also:

$$Au = \alpha u - \omega v$$
$$Av = \omega u + \alpha v$$

Gem. Basis (v, u) hat A die Gestalt

$$A = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}. \quad \square$$

Dann gilt

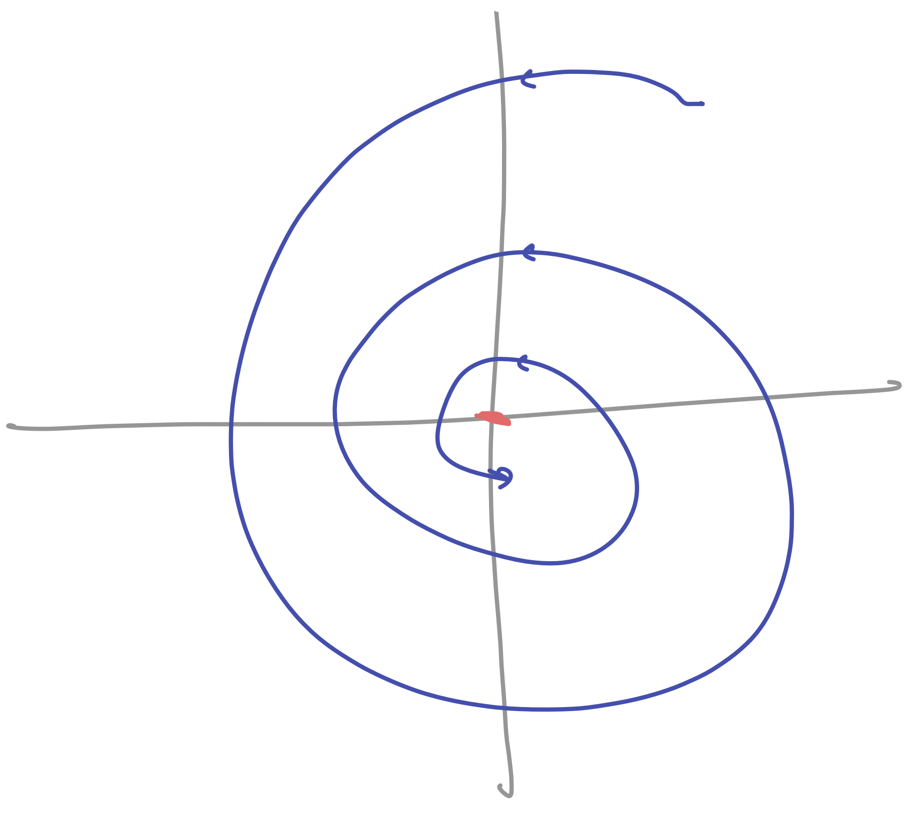
$$e^{At} = e^{\alpha I t + \omega J t}$$

$$= e^{\alpha I t} \underbrace{e^{\omega J t}}$$

$$= \underbrace{e^{\alpha t}}_{\text{Streckung}} \underbrace{\begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}}_{\text{Drehung um } \omega t}$$

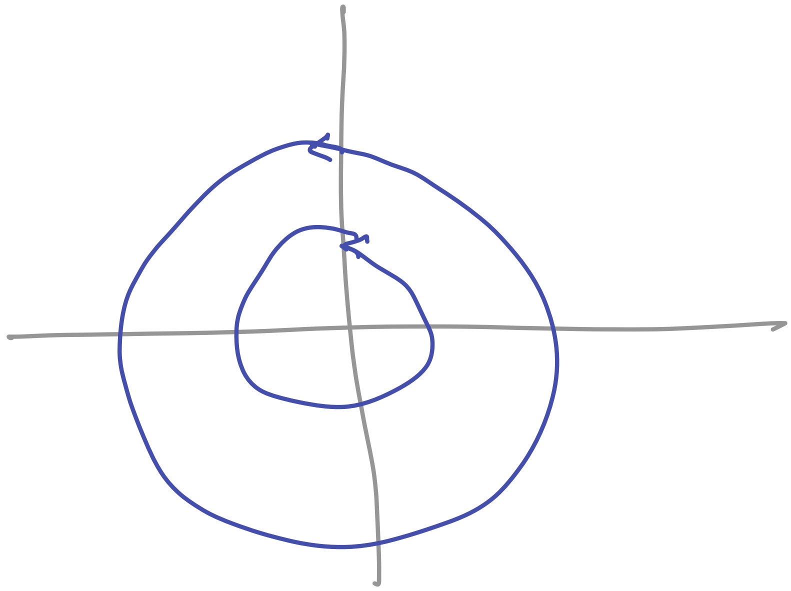
$$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Drehstreckung}}$$

$\alpha < 0$



stabiler Strudel

$\alpha = 0$



Zentrum