21. Vorlesung 6.7.2021

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= \lambda^2 - \lambda \underbrace{(a+d)}_{s := \operatorname{sp} A} + \underbrace{ad - bc}_{\det A =: d}$$

$$= \lambda^2 - s\lambda + d$$

EW:

$$\lambda_\pm = \frac{s \pm \sqrt{s^2 - 4d}}{2}$$

$$\lambda_\pm = \frac{S \pm \sqrt{S^2 - 4d}}{2}$$

$\det A < 0$: reell, verschiedene Vorz.:

$\longleftrightarrow$ Sattel

$\det A \geq 0$: Diskriminante

$$\Delta = S^2 - 4d$$

$\Delta > 0$: $\longleftrightarrow$ Knoten

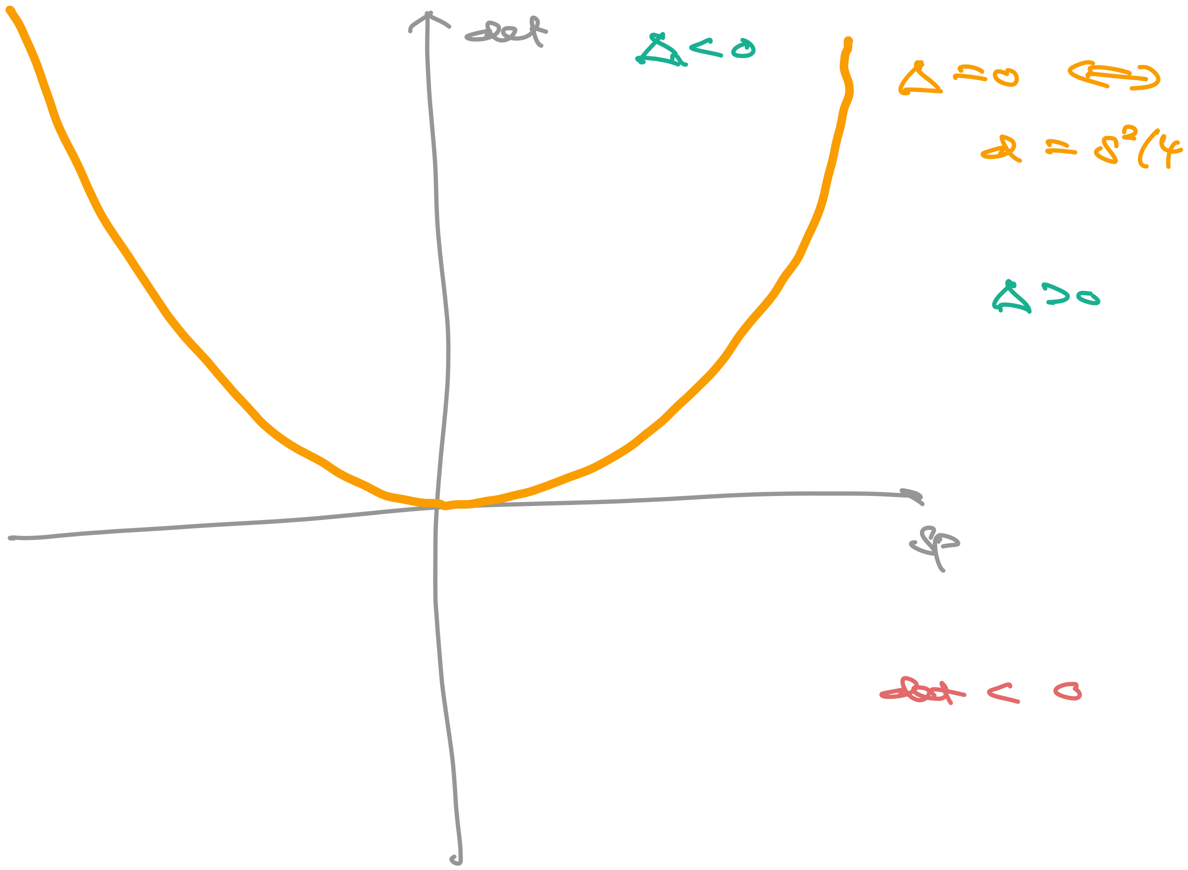
$\Delta = 0$: $\longleftrightarrow$ entartete Knoten

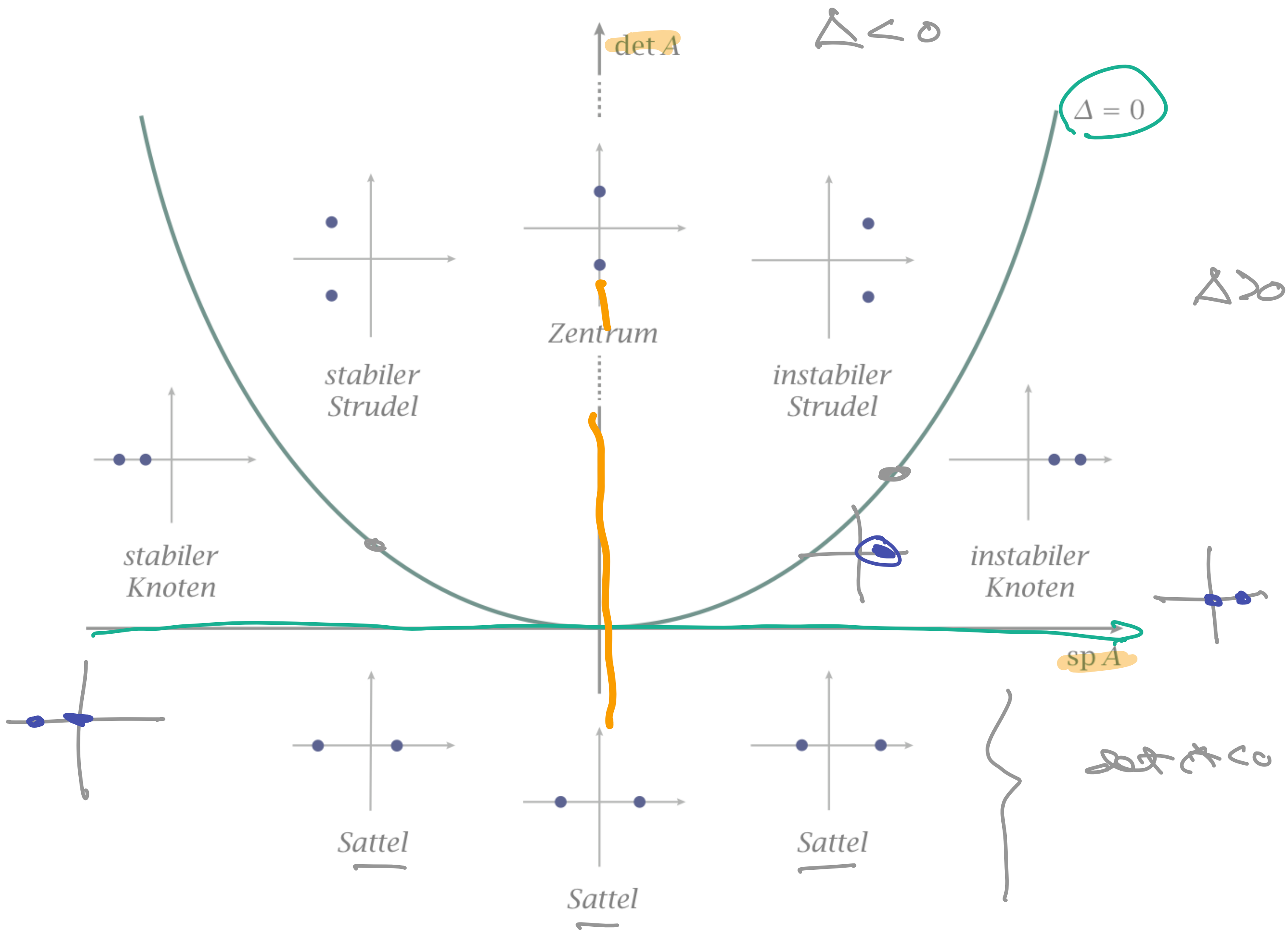
$\Delta < 0$: $\longleftrightarrow$ Strudel oder Zentrum

Stabil / Instabil :

$\operatorname{sp} A < 0$: stabil

$\operatorname{sp} A > 0$: instabil


det
$\Delta < 0$
$\Delta = 0 \iff$
$d = s^2/4$
$\Delta > 0$
tr
det < 0



$\det A = 0$ ⟿ A singulär

$\Delta = 0$ ⟿ Parabel : doppelt

$\operatorname{sp} A = 0$ ⟿ Zeit.-t-Achse periodisch

$$\boxed{\ddot{u} = -\omega^2 u - \rho \dot{u}}$$

$\omega > 0$

$\rho \geq 0$

$$x_1 = u$$
$$x_2 = \dot{u}$$

Dann

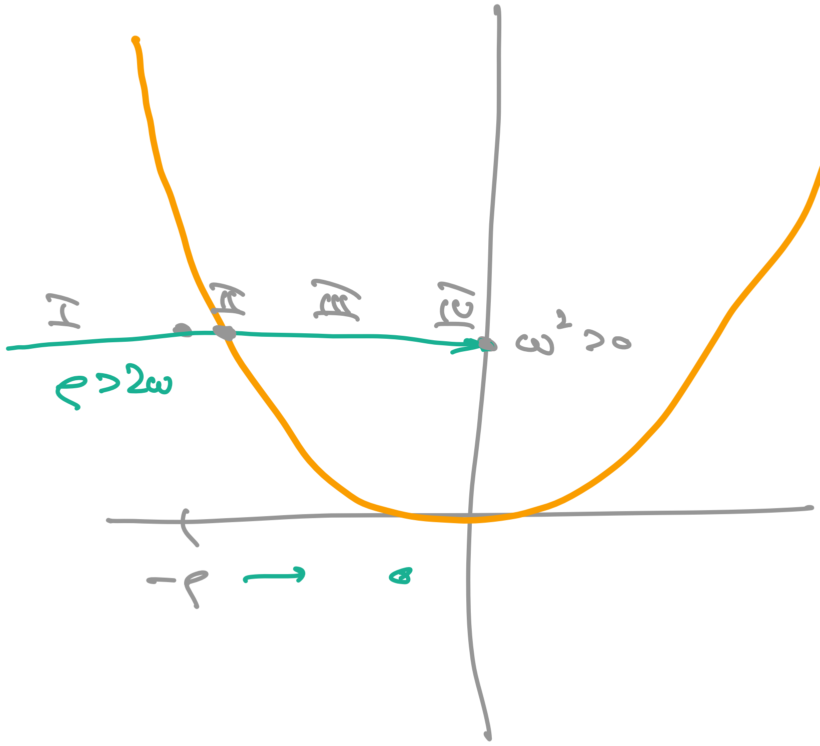
$$\dot{x}_1 = x_2$$
$$\ddot{u} = \dot{x}_2 = -\omega^2 x_1 - \rho x_2$$

Also:

$$\boxed{\dot{x} = Ax, \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\rho \end{pmatrix}}$$

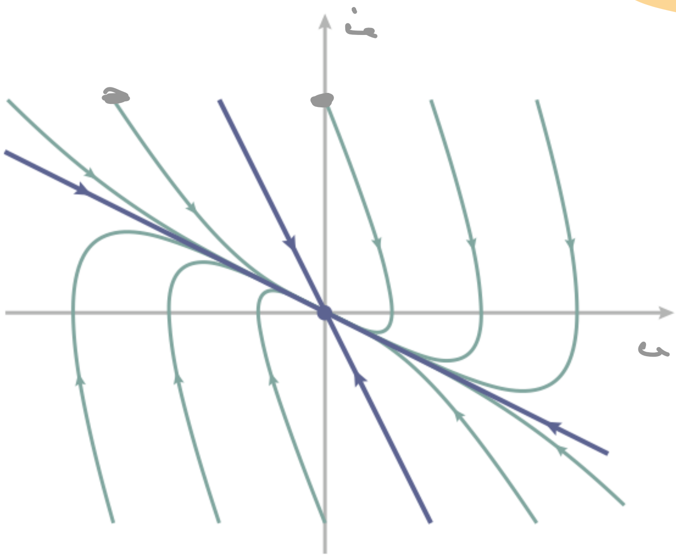
$\Delta = 0$

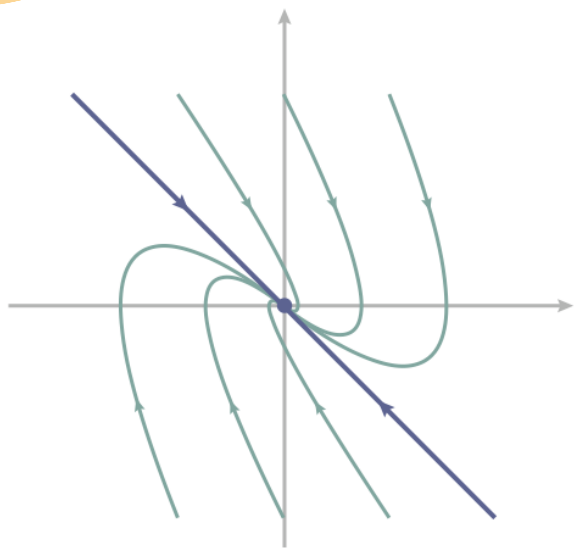
$\Leftrightarrow \quad \rho = 2\omega$

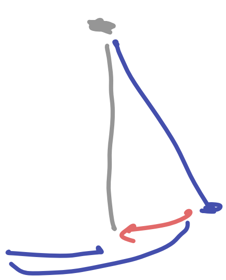


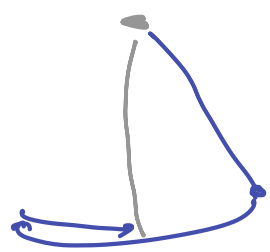
Stabiler Knoten für $\rho > 2\omega$ *:*

Entarteter stabiler Knoten für $\rho = 2\omega$ *:*







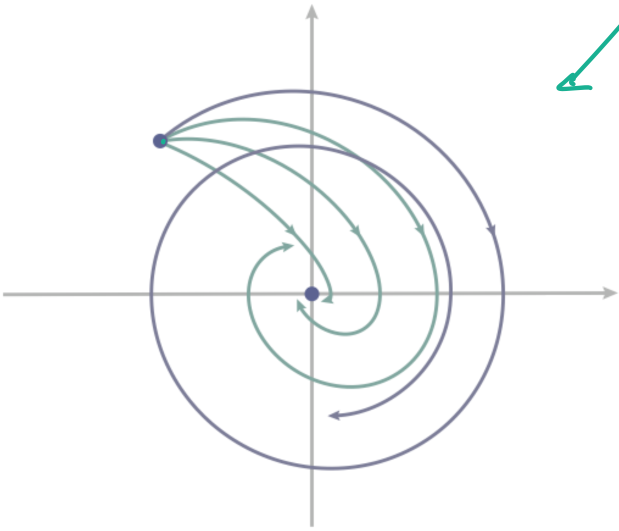


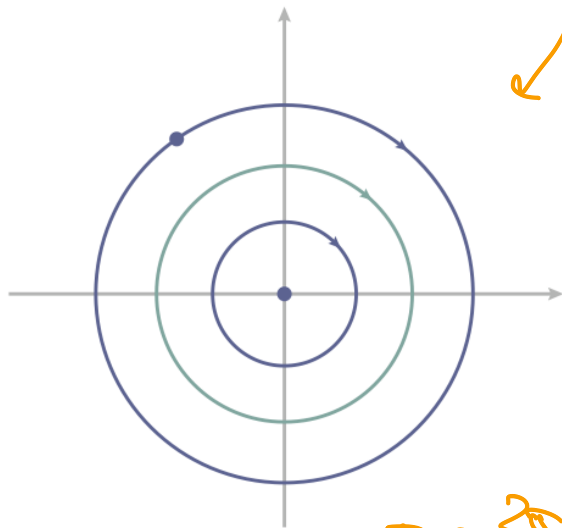
Stabiler Strudel für $0 < \rho < 2\omega$ *:*

gedämpfter Schw.

Zentrum für $\rho = 0$ *:*

ungedämpfte Schw.





$T = \frac{2\pi}{\omega}$.

Periode:

$\omega T = 2\pi$

$0 < \rho < 2\omega$:

Eigenwerte $\alpha \pm i\mu$ mit

$$\alpha = -\frac{\rho}{2}, \quad \mu = \sqrt{\omega^2 - \rho^2/4}$$

Fundamentalsystem :

$$e^{\alpha t} \cos \mu t, \qquad e^{\alpha t} \sin \mu t$$

Also:

$$u(t) = a e^{\alpha t} \cos \mu t + b e^{\alpha t} \sin \mu t$$

$r^2 = a^2 + b^2$:

$$= r e^{\alpha t} \left(\frac{a}{r} \cos \mu t + \frac{b}{r} \sin \mu t \right)$$

$$= r e^{\alpha t} \cos(\mu t + \tau)$$

Schwingung :

Amplitude $r e^{\alpha t}$

~~Phase~~ τ

Frequenz $\underbrace{\mu = \sqrt{\omega^2 - \rho^2/4}}_{\text{reduzierte Frequenz.}}$

Periode $T = \frac{2\pi}{\mu} \rightarrow \begin{cases} \infty, & \rho \to 2\omega \\ \frac{2\pi}{\omega}, & \rho \to 0 \end{cases}$

$$\dot{x} = Ax, \qquad x \in V,$$

$$\dim V = n \geq 2.$$

Basis:

$$Av = \lambda v$$

$$\begin{aligned}
(e^{\lambda t} v)^{\cdot} &= (e^{\lambda t})' \, v \\
&= e^{\lambda t} \cdot (\lambda v) \\
&= e^{\lambda t} \, A v \\
&= A (e^{\lambda t} v)
\end{aligned}$$

Geometrisch:

$$\operatorname{span}(v) \subset U$$

$$\dot{x} = \lambda x$$

λ reell: $\quad \dim \operatorname{span}(v) = 1$

kpx: $\quad = 2$

Sind $v_1, \dots, v_n$ Basis aus EV zu $\lambda_1, \dots, \lambda_n$, so bilden

$$\varphi_k(t) = e^{\lambda_k t} v_k, \quad 1 \le k \le n$$

ein <u>Fundamentalsystem</u> von $\dot{x} = Ax$.

$$T^{-1} A T = D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

$$\iff \quad AT = TD$$

Aus: Spalten von T sind die Eig.-v. von A

Allg. Lsg

$$\varphi(t) = T e^{Dt} c \quad , \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$$

$$Aw = \lambda w \qquad\qquad e^{\lambda t} w$$

$$\varphi(t) = e^{\lambda t} w$$

$$\overline{\varphi(t)} = e^{\bar{\lambda} t} \bar{w}$$

$$\lambda = \alpha + i\omega, \qquad w = v + iu \quad :$$

$$\begin{aligned} \varphi(t) &= e^{\lambda t} w \\ &= e^{\alpha t} (\cos \omega t + i \sin \omega t)(v + iu) \\ &= e^{\alpha t} \big((\cos \omega t) \cdot u - (\sin \omega t) v \big) \\ &\quad + i \ \dots \end{aligned}$$

$$A = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} \simeq \alpha + i\omega$$

$$X = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & \begin{pmatrix} \alpha_{r+1} & -\omega_{r+1} \\ -\omega_{r+1} & \alpha_{r+1} \end{pmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{pmatrix} \alpha_m & -\omega_m \\ \omega_m & \alpha_m \end{pmatrix} \end{pmatrix}$$

Bsp: $\dot{x} = Ax$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix}$$

Ew:

$1 : \begin{pmatrix} -10 \\ 3 \\ 1 \end{pmatrix} = w$

$2 \pm 3i : \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$= v + i u$

Die allgemeine Lösung:

$$\varphi(t) = c_1 e^{t} w + e^{2t}(c_2 \sin 3t + c_3 \cos 3t)\, v + e^{2t}(c_2 \cos 3t - c_3 \sin 3t)\, u$$

Transformatie :

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$x = Ty$:

$$T^{-1}AT = \begin{pmatrix} 1 & & \\ & 2 & -3 \\ & 3 & 2 \end{pmatrix} = B$$

$$e^{Bt} = \begin{pmatrix} e^{t} & & \\ & e^{2t}\cos 3t & -e^{2t}\sin 3t \\ & e^{2t}\sin 3t & e^{2t}\cos 3t \end{pmatrix} \begin{pmatrix} c \\ c \\ c \end{pmatrix}$$

$=$