3. Vorlesung 26.4.2021

$$F' = f$$

Allg:

$$F'_- = f_- , \quad F'_+ = f_+$$

auf ganz $[a,b]$

Beweis: Ist f auf I integrierbar, dann auch auf jedem Teilintervall I.

Dann ist F wohldefiniert.

$$F : I \longrightarrow \mathbb{R} .$$

Bew Lipschitzstetig:

$$(*) \quad \Phi(v) - \Phi(u) = \int_c^v f - \int_c^u f = \int_u^v f,$$

$u, v \in I.$

Für $u < v$:

$$\underbrace{|\Phi(u) - \Phi(v)|} = \left|\int_u^v f\right| \underset{u \le v}{\le} \int_u^v |f| \le \underbrace{\|f\|_I \cdot (v-u)}_{\text{Lipschitz.}}$$

Setze $u = t$, $v = t+h$, :

$$\boxed{\frac{\Phi(t+h) - \Phi(t)}{h} = \frac{1}{h} \cdot \int_t^{t+h} f.}$$

Riemann-Lemma:

$$\boxed{\lim_{h \downarrow 0} \frac{\Phi(t+h) - \Phi(t)}{h} = \lim_{h \downarrow 0} \frac{1}{h} \cdot \int_t^{t+h} f.}$$

$$= f_+(t)$$

D.C.

Also: Φ ist bei t rechtsseitig diffbar,

$$\Phi_+'(t) = f_+(t)$$

Also ist Φ Stammf. zu f. $\square$

Beweis: Sei $t \in [a,b)$, $h > 0$ so klein, dass $t+h \in (a,b]$. Dann gilt:

$$f_+(t) = f_+(t) \cdot \underbrace{\frac{1}{h}\int_t^{t+h} 1}_{=1}$$

$$= \frac{1}{h}\int_t^{t+h} f_+(t)$$ Konstant bzgl. Integration

Also:

$$\left| f_+(t) - \frac{1}{h}\int_t^{t+h} f \right|$$

$$= \left| \frac{1}{h}\int_t^{t+h} f_+(t) - \frac{1}{h}\int_t^{t+h} f \right|$$

$$= \left| \frac{1}{h}\int_t^{t+h} \left(f_+(t) - f\right) \right|$$

$$\underset{h>0}{\le} \frac{1}{h}\int_t^{t+h} \left| f_+(t) - f \right| .$$

Zu jedem $\varepsilon > 0$ ex. $\delta > 0$:

$$\left| f_+(t) - f(s) \right| < \varepsilon, \quad t < s < t+\delta$$

Also gilt für $h < \delta$:

$$\dots \le \frac{1}{h} \cdot \int_t^{t+h} \varepsilon = \varepsilon$$

Da [illegible] [illegible] $\varepsilon > 0$ .....

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f = f_+(t) \,, \qquad \blacksquare$$

Beispiele.

1. sgn : Dann

$$F(t) = \int_0^t \operatorname{sgn}$$

$$= \begin{cases} \int_0^t 1, & t \geq 0 \\ \int_0^t -1, & t < 0 \end{cases}$$

$$= \begin{cases} t, & t \geq 0 \\ -t, & t < 0 \end{cases}$$

$$= |t| \,.$$

2. $$\int_0^t |s| = \frac{1}{2} t \cdot |t|$$

$$= \frac{1}{2} t^2 \cdot \operatorname{sgn}(t)$$

$$= \begin{cases} \frac{1}{2} t^2, & t \geq 0 \\ -\frac{1}{2} t^2, & t < 0 \end{cases} .$$

$\square$

$$\int_a^b f = F\Big|_a^b := F(b) - F(a)$$

Fie [illegible] [illegible] [illegible] a f.

Dem: Fie [illegible]

$$\Phi(x) = \int_c^x f .$$

Dem

$$\Phi(b) - \Phi(a) = \int_c^b f - \int_c^a f$$

$$= \int_a^b f .$$

Fie [illegible] [illegible] [illegible] F [illegible] [illegible]:

$$F = \Phi + c .$$

Dem [illegible] [illegible]

$$F(b) - F(a) = \Phi(b) - \Phi(a) = \int_a^b f \quad \blacksquare$$

Bsp:

1. Sei $|\cdot|$ Stammf. für sgn:

$$\int_a^b \operatorname{sgn} = |t| \Big|_a^b = |b| - |a| .$$

2. $$\int_1^4 \frac{1}{\sqrt{t}} = 2\sqrt{t} \Big|_1^4$$

$$= 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2 .$$

Spezialfall: f C^1-Funktion:

f Stammf. von f':

$$\int_a^b f' = f \Big|_a^b$$

Anwendung auf 2: $$2\sqrt{t} \Big|_1^4 = \int_1^4 (2\sqrt{t})' = \int_1^4 \frac{1}{\sqrt{t}} .$$

[illegible]:

$$\int f = F + c$$

Bsp:

$$\int e^t = e^t + c$$

$$\int \frac{1}{t} = \ln t + c, \quad t > 0$$

$$\int \cos t = \sin t + c$$

$$\int \frac{1}{\sqrt{1-t^2}} = \arcsin t + c, \quad -1 \le t \le 1.$$

$$\int_a^b f = \int_a^b f(t)\,dt$$

$$= \int_a^b f(u)\,du = \int_a^b f(x)\,dx$$

<u>Bew:</u> Die Integrale existieren, da f, g C^1.

$$\int_a^b f'(t)\,g(t)\,dt + \int_a^b f(t)\,g'(t)\,dt$$

$$= \int_a^b \left(f'(t)\,g(t) + f(t)\,g'(t) \right) dt$$

$$= \int_a^b \left(f(t)\,g(t) \right)'\,dt$$

$$= f(t)\,g(t) \Big|_a^b . \qquad \blacksquare$$

$$\boxed{\int \overset{F'}{\overset{\|}{f(t)}}\,g(t)\,dt = F(t)\,g(t) - \int F(t)\,g'(t)\,dt}$$

Bsp:

1. $$\int t \underset{f}{e^t}\, dt = t \cdot \underbrace{e^t}_{F} - \int (t)' e^t$$

$$= t e^t - \int e^t$$

$$= t e^t - e^t + c$$

$$= (t-1) e^t + c$$

2. B.

$$\int_0^1 t e^t\, dt = (t-1) e^t \Big|_0^1$$

$$= 1.$$

2. $\int \cos^2 t \, dt$

$$= \int \cos t \cdot \cos t \, dt$$

$$= \sin t \cdot \cos t - \int \sin t \cdot (\cos t)' \, dt$$

$$= \sin t \cdot \cos t + \int \underbrace{\sin^2 t}_{\substack{\| \\ 1 - \cos^2 t}} \, dt$$

$$= \underbrace{\sin t \cdot \cos t + t}\ \underbrace{- \int \cos^2 t \, dt}$$

Also:
$$\boxed{\int \cos^2 t \, dt = \frac{1}{2} \left(\sin t \cdot \cos t + t \right)}$$

$$\int_0^{\pi/2} \cos^2 t \, dt = \frac{1}{2} \left(\sin t \cdot \cos t + t \right) \Big|_0^{\pi/2} = \frac{\pi}{4}.$$

Das geht auch einfacher: Aus Symmetriegründen:

$$\int_0^{\pi/2} \cos^2 t \, dt = \frac{1}{4} \int_0^{\pi} \big(\underbrace{\cos^2 t + \sin^2 t}_{1} \big) \, dt = \frac{\pi}{4}.$$

3. $\int \underbrace{\log t}\, dt$

$= \int 1 \cdot \log t \, dt$

$= t \cdot \log t - \int t \cdot (\log t)' \, dt \qquad = \frac{1}{t}$

$= t \cdot \log t - \int \underbrace{t \cdot \frac{1}{t}}_{1} \, dt$

$= t \cdot \log t - t \quad + c$

$\int_1^e \log t \, dt = \left(t \cdot \log t - t\right)\Big|_1^e$

$= e - e - (-1) = 1.$

Beweis: Per Induktion über n:

$n=0$:

$$f(x+h) = f(x) + R_x^0 f(h)$$

$$= f(x) + \underbrace{h \cdot \int_0^1 f'(x+th)\,dt}$$

$$= \int_0^1 \left(f(x+th)\right)'\,dt$$

$$= f(x+th)\Big|_0^1$$

$$= f(x+h).$$

$$\frac{h^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n+1)}(x+th)\,dt$$

$$= \frac{h^{n+1}}{n!}\left(-\frac{(1-t)^{n+1}}{n+1} f^{(n+1)}(x+th)\right)\Big|_0^1$$

$$+ \frac{h^{n+1}}{n!}\,\frac{1}{n+1} \int_0^1 (1-t)^{n+1} f^{(n+2)}(x+th)\cdot h\,dt$$

$$= \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(x) + \underbrace{\frac{h^{n+2}}{(n+1)!} \int_0^1 (1-t)^{n+1} f^{(n+2)}(x+th)\,dt}_{R_x^{n+1} f(h)}$$

$\blacksquare$

$$f(x+h) - f(x)$$

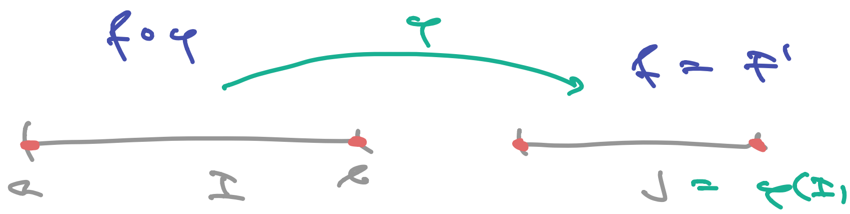
$$= f(x+th)\Big|_0^1$$

$$= \int_0^1 \left(f(x+th) \right)' \, dt$$

$$= h \cdot \int_0^1 f'(x+th) \, dt$$

Kompakt

$$f(x+h) = f(x) + h \cdot \int_0^1 1 \cdot f'(x+th) \, dt$$



Beweis: Sei F Stammfunktion von f.

Dann gilt:

$$(F \circ \varphi)' = (F' \circ \varphi)\,\varphi' = (f \circ \varphi) \cdot \varphi' .$$

[illegible]:

$$\boxed{\int_a^b f(\varphi(s)) \cdot \varphi'(s)\, ds} = \int_a^b (F \circ \varphi)'(s)\, ds$$

$$= (F \circ \varphi) \Big|_a^b$$

$$= F \Big|_{\varphi(a)}^{\varphi(b)} \boxed{= \int_{\varphi(a)}^{\varphi(b)} f(t)\, dt}$$

$\blacksquare$