4. Vorlesung 27. 4. 2021

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$$

$$= F(t)\Big|_{\varphi(a)}^{\varphi(b)}$$

Bsp:

$2t = (1+t^2)'$

1.
$$\int \frac{t}{\sqrt{1+t^2}}\,dt$$

$$= \frac{1}{2}\int \frac{2t}{\sqrt{1+t^2}}\,dt$$

$f = \frac{1}{\sqrt{t}}$

$\varphi = 1+t^2$

$F = 2\sqrt{t}$

$$= \frac{1}{2}\; 2\sqrt{1+t^2} + c$$

$$= \sqrt{1+t^2} + c.$$

$$\int_a^b \frac{t}{\sqrt{1+t^2}}\,dt = \sqrt{1+t^2}\,\Big|_a^b \;.$$

2. $\displaystyle\int \frac{\log t}{t}\,dt$ $\qquad \frac{1}{t} = (\log t)'$

$= \displaystyle\int (\log t)\cdot(\log t)'\,dt$

$x = \log t$

$f = x$

$F = \frac{1}{2}x^2$

$= \frac{1}{2}x \Big|_{x=\log t} + c$

$= \frac{1}{2}\log^2 t + c$

3. $\displaystyle\int \tan t\,dt$

$= \displaystyle\int \frac{\sin t}{\cos t}\,dt$ $\qquad \sin t = -(\cos t)'$

$f = \frac{1}{x}$

$\cos t = x$

$= -\displaystyle\int \frac{(\cos t)'}{\cos t}\,dt$

$F = \log x$

$= -\log(\cos t) + c$

$= \log \frac{1}{\cos t} + c.$ $\qquad \blacksquare$

$t = \varphi(s)$

donc

$$\boxed{\frac{dt}{ds} = \frac{d(\varphi(s))}{ds} = \varphi'(s)}$$

donc

$$\boxed{dt = \varphi'(s)\, ds}.$$

$\varphi(a) = \alpha$

$\varphi(b) = \beta$

Παρ:

$$\int t\sqrt{1+t}\,dt \qquad t = \varphi(x)$$

Προσοχή: $\quad 1+t = x^2 \quad , \quad t = x^2 - 1$

$$dt = 2x \cdot dx$$

$$\ldots = \int (x^2-1)\sqrt{x^2}\; 2x\,dx$$

$$= \int (x^2-1)\, x \cdot 2x\,dx$$

$$= 2\int (x^4 - x^2)\,dx\,.$$

[illegible]: $\quad t = t_0 \rightsquigarrow x_0 = \sqrt{1+t_0}$

$$\int_a^b t\sqrt{1+t}\,dt = 2\int_{\sqrt{1+a}}^{\sqrt{1+b}} (x^4 - x^2)\,dx = \ldots$$

2. $\displaystyle\int_{-1}^{1} \sqrt{1-t^2}\, dt$

$t = \sin x$

$dt = (\sin x)'\, dx = \cos x\, dx$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\underbrace{1-\sin^2 x}_{\cos^2 x}} \cdot \cos x\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos x \cdot \cos x\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = \frac{\pi}{2}.$$

Ebenso:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\, dt = \int_{-\pi/2}^{\pi/2} \cancelto{1}{\frac{\cos x}{\cos x}}\, dx = \pi.$$

Oder:

$$= \arcsin t \Big|_{-1}^{1} = \pi.$$

3.

$$\int_a^b f(t+c)\,dt = \int_{a+c}^{b+c} f(x)\,dx$$

$x = t+c$

$$\int_a^b f(\lambda t)\,dt = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f(x)\,dx$$

$\lambda t = x$

$\lambda \neq 0$

$x = t^n$:

$$\int_a^b f(t^n)\,t^{n-1}\,dt = \frac{1}{n}\int_{a^n}^{b^n} f(x)\,dx \quad ,$$

$n \geq 1$.

$\blacksquare$

Betrachte

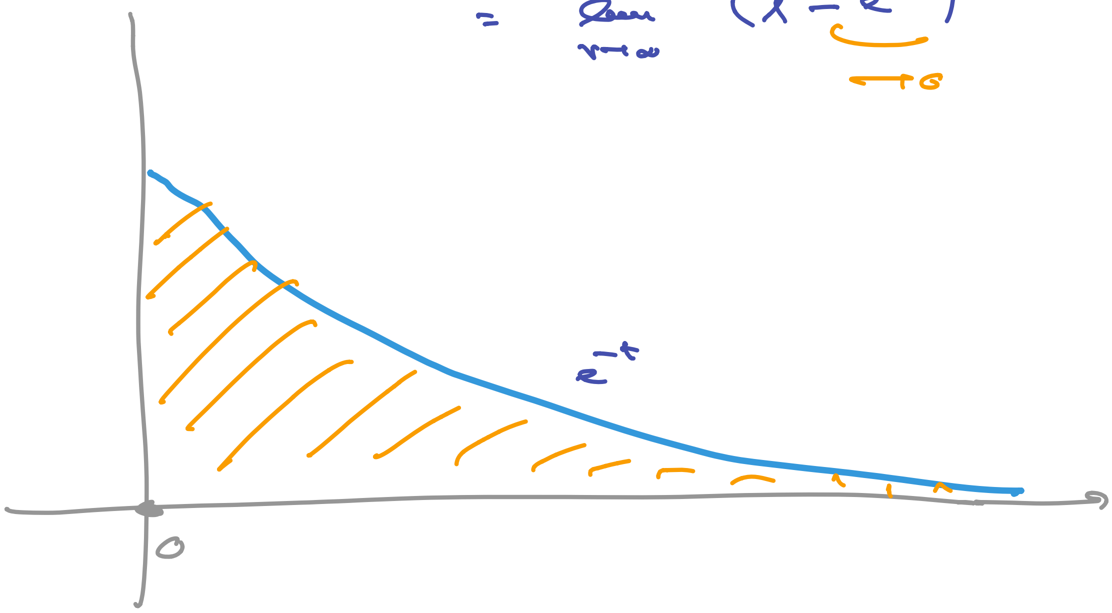
$f: \quad [a,b) \longrightarrow \mathbb{R}, \quad a < b \leq \infty$

Bsp:

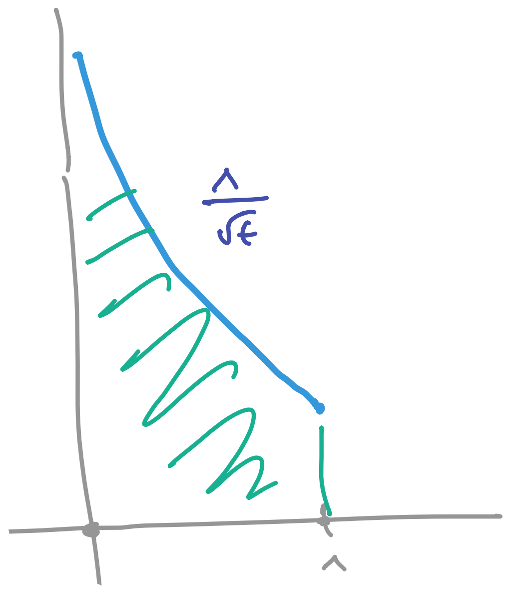
1. $$\int_0^\infty e^{-t}\,dt = \lim_{r\to\infty}\int_0^r e^{-t}\,dt$$

$$= \lim_{r\to\infty}\left(-e^{-t}\Big|_0^r\right)$$

$$= \lim_{r\to\infty}\left(1-\underbrace{e^{-r}}_{\to 0}\right) = 1.$$



2. $$\int_0^1 \frac{1}{\sqrt{t}}\,dt$$

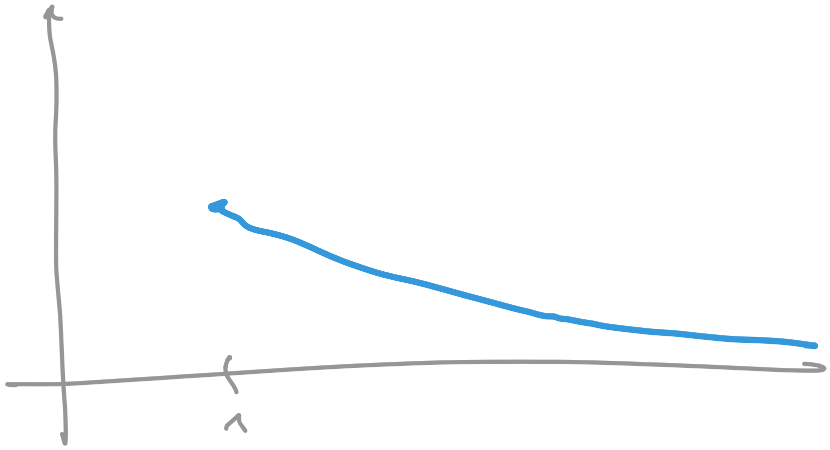


$$= \lim_{\varepsilon\searrow 0}\int_\varepsilon^1 \frac{dt}{\sqrt{t}}$$

$$= \lim_{\varepsilon\searrow 0}\left(2\sqrt{t}\Big|_\varepsilon^1\right)$$

$$= \lim_{\varepsilon\searrow 0}\left(2-2\sqrt{\varepsilon}\right) = 2.$$

3. $\int_1^\infty \frac{dt}{\sqrt{t}}$



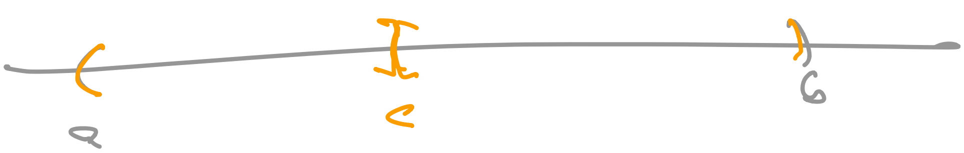
$$= \lim_{r \to \infty} \int_1^r \frac{dt}{\sqrt{t}}$$

$$= \lim_{r \to \infty} \left(2\sqrt{t} \Big|_1^r \right)$$

$$= \lim_{r \to \infty} \left(2\sqrt{r} - 2 \right) = \infty .$$

Zz: $(a,b) = (-\infty, \infty) = \mathbb{R}$

$[c,d] \subset \mathbb{R}$



Bsp:

1. $$\int_{-\infty}^{\infty} |t|\, e^{-t^2}\, dt$$

$t \geq 0$:

$$\int_0^{\infty} t\, e^{-t^2}\, dt = \lim_{r \to \infty} \int_0^r t\, e^{-t^2}\, dt$$

$$= \lim_{r \to \infty} \left(-\frac{1}{2} e^{-t^2}\right)\Big|_0^r$$

$$= \frac{1}{2}.$$

Analog:

$$\int_{-\infty}^{0} (-t)\, e^{-t^2}\, dt = \frac{1}{2}$$

Also:
$$\int_{-\infty}^{\infty} |t|\, e^{-t^2}\, dt = 1.$$

Bem: $c_1, c_2 \in (a,b)$:

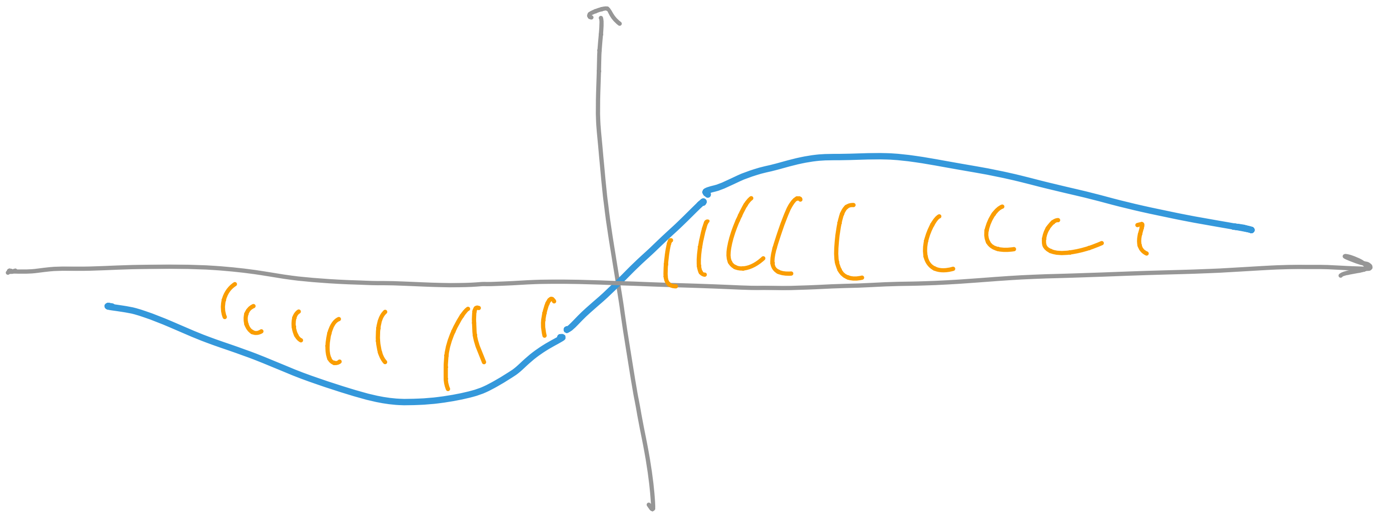
$$\int_a^{c_1} + \int_{c_1}^{b} = \int_a^{c_2} + \int_{c_2}^{b}$$

Denn: $\int_{c_1}^{c_2} f$ ex. und ist endlich

Bsp:

2. Betrachte

$$\int_{-\infty}^{\infty} \frac{t}{1+t^2}\, dt$$



Es gilt:

$$\int_{-r}^{r} \frac{t}{1+t^2}\, dt = \frac{1}{2} \log(1+t^2) \Big|_{-r}^{r} = 0$$

$$\lim_{r \to \infty} \int_{-r}^{r} \cdots = 0.$$

$$\text{"}\cancel{\infty - \infty = 0}\text{"}$$

$c = 0$:

$$\int_0^u \frac{t}{1+t^2}\,dt = \frac{1}{2}\log(1+t^2)\Big|_0^u$$

$$= \frac{1}{2}\log(1+u^2)$$

$$\longrightarrow \infty \qquad \lightning$$

Also:

$$\int_0^\infty \frac{t}{1+t^2}\,dt = \infty$$

Bew: Per Definition:

$$\lim_{c\uparrow b} \int_a^c f(t)\,dt = \lim_{c\uparrow b} F(t)\Big|_a^c$$

$$= \lim_{c\uparrow b} \big(F(c) - F(a)\big)$$

$$= \Big(\lim_{c\uparrow b} F(c)\Big) - F(a).$$

Beweis: Betrachte

$$F(t) = \int_a^t |f(s)|\, ds .$$

Stammfunktion von $|f|$ auf $[a,b)$.

Da $|f| \geq 0$: ist F monoton steigend auf $[a,b)$. Dann:

$\lim_{x \to b} F(x)$ existiert genau dann, wenn F auf $[a,b)$ beschränkt ist.

Also wenn $\int_a^b |f(s)|\, ds$ existiert.

Dreiecksungleichung:

$$\left| \int_u^v f(t)\,dt \right| \leq \int_u^v |f(t)|\,dt, \quad u<v \text{ in } [a,b)$$

Damit: absolute Konv. $\Rightarrow$ einfache Konv. ∎

Beweis: Für $\alpha \neq 1$:

$$\underset{\neq 0}{(1-\alpha)} \int_1^r \frac{dt}{t^\alpha} = (1-\alpha) \int_1^r t^{-\alpha}\, dt$$

$$= t^{-\alpha+1} \Big|_1^r$$

$$= r^{1-\alpha} - 1 .$$

Jetzt: $r \to \infty$: $\qquad$ $r \to 0$

$r^{1-\alpha}$ konvergiert $\iff 1-\alpha < 0$ $\qquad$ $1-\alpha > 0$

$\iff \alpha > 1$. $\qquad$ $\iff \alpha < 1$.

Für $\alpha = 1$: $\displaystyle\int_1^r \frac{dt}{t} = \log r \longrightarrow \pm\infty$

$r \to \infty$

$r \to 0$

Für $\displaystyle\int_0^1 \frac{dt}{t^\alpha}$: $\quad$ S.o.: für $0 < r < 1$,

$r \to 0$

Bsp:

1. $$\int_1^\infty \frac{\sin t}{t^\alpha}\,dt$$

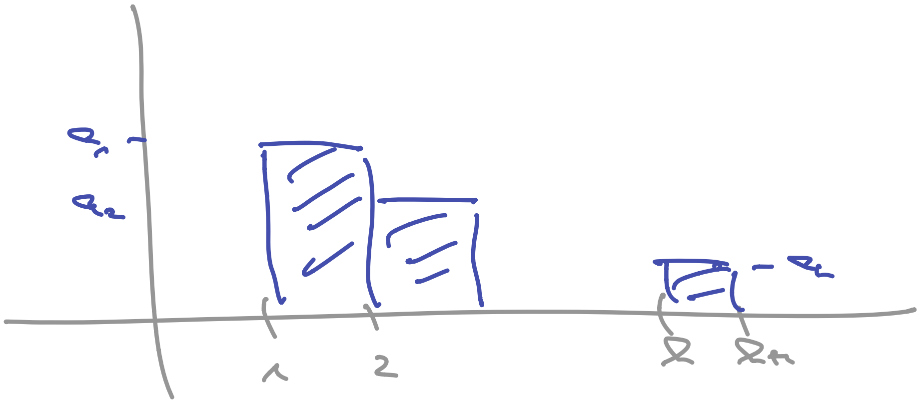
ist absolut konvergent für $\alpha > 1$:

$$\left|\frac{\sin t}{t^\alpha}\right| \leq \frac{1}{t^\alpha}$$

2. $$\int_1^\infty \frac{\sin t}{t}\,dt \qquad ?$$

L: ist konvergent, aber nicht absolut konvergent. $\square$

$$\sum_{n=1}^{\infty} a_n = \int_1^{\infty} a(t)\,dt$$



$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a(n)$$

$$\underbrace{a(t)}_{\text{monoton fallend} \downarrow}$$

$$\lim_{t \to \infty} a(t) = 0.$$

Beweis: Da $a \geq 0$:

$$s(n) = \sum_{k=1}^{n} a(k) \quad , \quad S(t) = \int_1^t a(x)\,dx$$

monoton wachsend

so konvergent $\Longleftrightarrow$ beschränkt.

Da a monoton fällt:

$$a(k-1) \geq a(t) \geq a(k) \quad , \quad k-1 \leq t \leq k, \; k \geq 2$$

Also: $\int_{k-1}^{k}$

$$a(k) \leq \int_{k-1}^{k} a(t)\,dt \leq a(k-1) \quad , \quad \text{---} \text{---}$$

$\sum_{k=2}^{n}$

Also:

$$\boxed{s(n) - a(1) \leq \int_1^n a(t)\,dt \leq s(n-1)}$$

Daraus folgt die Behauptung. $\square$

Bsp.:

$$\sum_{n\geq 2} \underbrace{\frac{1}{n\cdot \log n}}_{\searrow \text{ in } n} = \infty$$

$$\int_2^\infty \frac{dt}{t\cdot \log t} = \int_2^\infty (\log t)' \frac{dt}{\log t}$$

$$= \log(\log t)\Big|_2^\infty \longrightarrow \infty$$

Aber: $\varepsilon > 0$:

$$\int_2^\infty \frac{dt}{t\cdot \log^{1+\varepsilon} t} = -\frac{1}{\varepsilon \log^{\varepsilon} t}\Big|_2^\infty < \infty$$

Also:

$$\sum_{n\geq 2} \frac{1}{n \log^{1+\varepsilon} n} < \infty . \qquad \boxtimes$$