6. Vorlesung 4.5.2021

$\dot{x} = g(t)$ Stammf.

$\dot{x} = f(x)$ autonome G.

$f(x_0) = 0$, $x_0 \in J$

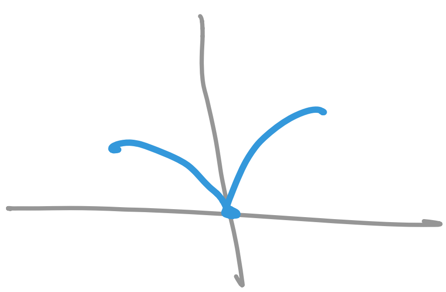
Dann

$x(t) \equiv x_0$, $t \in I$

Bsp:

$\dot{x} = 3x^{2/3}$, $x(0) = 0$

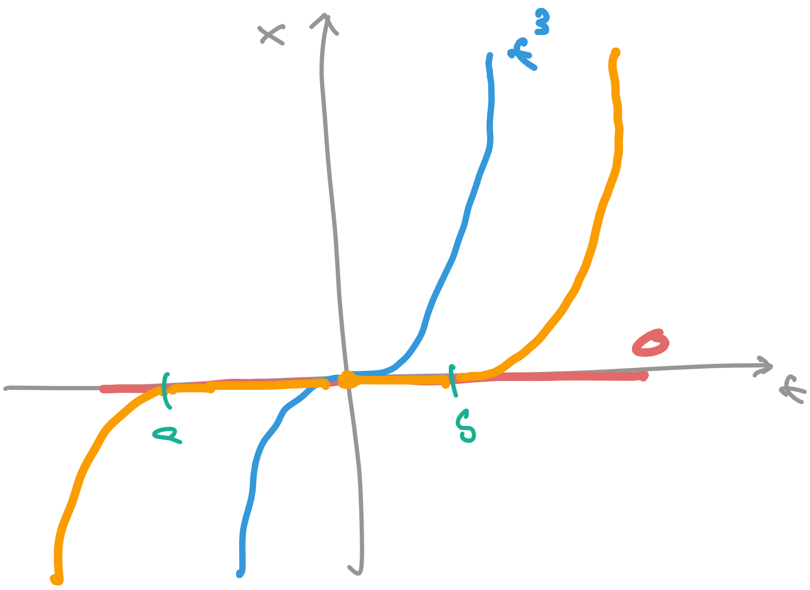
$= 3\sqrt[3]{x^2}$



1. $x(t) \equiv 0$. $x(t) = t^{\alpha}$?

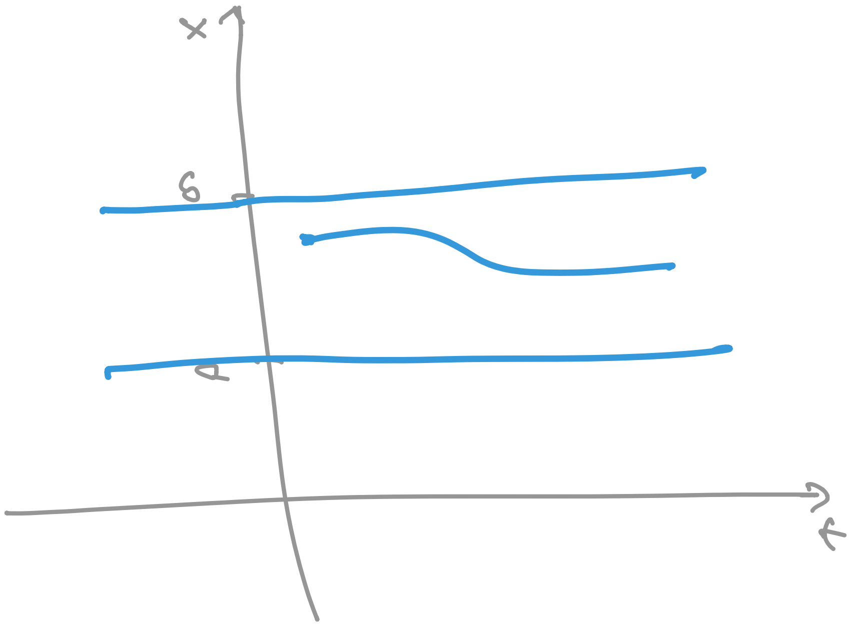
2. $x(t) = t^3$:

$$\dot{x}(t) = 3t^2 = 3(t^3)^{2/3} = 3\,x(t)^{2/3}.$$



$$x(t) = \underbrace{\begin{cases} (t-a)^3, & t < a \le 0 \\ 0, & a \le t \le b \\ (t-b)^3, & t > b \ge 0 \end{cases}}_{\infty \text{ viele Lsg}}$$

$$\dot{x} = g(t)\, f(x)$$



Also jetzt

$$\dot{x} = g(t)\, h(x), \qquad x \in J.$$

$$h(x) \neq 0 \quad \text{für} \quad x \in J.$$

Dann:

$$g(t) = \frac{\dot{\varphi}(t)}{h(\varphi(t))}$$

Dann:

$$\int_{t_0}^{t} g(s)\, ds = \int_{t_0}^{t} \frac{\dot{\varphi}(s)}{h(\varphi(s))}\, ds$$

$$= \int_{\varphi(t_0)}^{\varphi(t)} \frac{1}{h(u)}\, du$$

$$\boxed{\int_{t_0}^{t} g(s)\, ds = \int_{\varphi(t_0)}^{\varphi(t)} \frac{du}{h(u)}}.$$

Beweis:

Konstruktion:

$$\underline{G(t)} = \int_{t_0}^{t} g(s)\,ds = \int_{\varphi(t_0)}^{\varphi(t)} \frac{du}{h(u)} \qquad H' = \frac{1}{h}$$

$$= \underline{H(\varphi(t))}$$

Eindeutigkeit: Da $H' = \frac{1}{h} > 0$ auf J, ist H streng monoton.

Also: H ist invertierbar, also

$$\underbrace{\varphi(t) = H^{-1}(G(t))}.$$

Existenz: Nehme $\varphi(t)$:= Definition von φ.

Zu zeigen:

$$\varphi(t_0) = H^{-1}(G(t_0)) = H^{-1}(0) = x_0. \checkmark$$

Beweis:

$$G(t) = F(\varphi(t))$$

$$G'(t) = g(t) = F'(\varphi(t)) \cdot \dot\varphi(t) = \frac{\dot\varphi(t)}{h(\varphi(t))}$$

$$\Longleftrightarrow \quad \dot\varphi(t) = g(t) \cdot h(\varphi(t)). \quad ✓ \quad \square$$

Bem: Seien G und F Stammfunktionen obiger Gleichung, und

$$\Phi(t,x) = G(t) - F(x)$$

$$\Phi: I \times J \longrightarrow \mathbb{R}$$

Dann ist Φ konstant entlang jeder Lösungskurve.

Erhaltungsgröße = Integral

$$\frac{dx}{dt} = \dot{x} = g(t)\, h(x)$$

Dann:

$$\frac{dx}{h(x)} = g(t)\, dt$$

Kurzschreibweise für Integral:

$$\int \frac{dx}{h(x)} = \int g(t)\, dt \, .$$

Beispiele:

1. $\dot{x} = a(t)\, x$

Separation: $g(t) = a(t)$

$h(x) = x$

einzige NS von h: $x = 0$,

unabhängig der Separation — Eindeutigkeit

$$G(t) = \int g(s)\,ds = \int a(s)\,ds = A(t)$$

und

$$H(x) = \int \frac{dx}{h(x)} = \int \frac{dx}{x} = \log|x| + c$$

$(x \neq 0)$

Zur Lösung:

$$H(x) = G(t):$$

Ergibt:

$$A(t) = \log|x(t)| + c$$

Also:

$$\underline{|x(t)|} = e^{A(t)-c} = \underbrace{e^{A(t)}}_{>0} \cdot \underbrace{e^{c}}_{>0}, \quad c \in \mathbb{R}$$

Zusammen:

$$\underbrace{x(t) = e^{A(t)}\, c, \qquad c \in \mathbb{R}}_{\text{allgemeine Lösung.}}$$

2. Betrachte

$$\dot{x} = \frac{t}{x}, \qquad 0 < t, x < \infty .$$

$$\frac{dx}{dt} = \frac{t}{x},$$

also

$$x\,dx = t\,dt$$

$\int$:

$$\int_{t_0}^{t} s\,ds = \cancel{\tfrac{1}{2}}(t^2 - t_0^2) = \int_{x_0}^{x} u\,du = \cancel{\tfrac{1}{2}}(x^2 - x_0^2)$$

Also:

$$x(t)^2 = t^2 - t_0^2 + x_0^2$$

also:

$$x(t) = \sqrt{t^2 - t_0^2 + x_0^2} .$$

$$t > \begin{cases} 0, & t_0 \le x_0 \\ \sqrt{t_0^2 - x_0^2}, & \text{sonst.} \end{cases}$$

Homogen:

$$f(t,x) = h\left(\frac{x}{t}\right) :$$

Dann:

$$f(\lambda t, \lambda x) = f(t,x)$$

Complet:

$$f(t,x) = f\left(1, \frac{x}{t}\right) =: h\left(\frac{x}{t}\right)$$

Beweis: Sei φ Lösung von

$$\dot{x} = f\left(\frac{x}{t}\right),$$

und $\psi(t) = \frac{\varphi(t)}{t}$:

$$\dot{\psi}(t) = \left(\frac{\varphi}{t}\right)^{\cdot} = \frac{\dot{\varphi}}{t} - \frac{\varphi}{t^2}$$

$$= f\left(\frac{\varphi}{t}\right)\cdot\frac{1}{t} - \frac{\varphi}{t^2}$$

$$= \frac{f(\psi) - \frac{\varphi}{t}}{t}$$

$$= \frac{f(\psi) - \psi}{t}$$

Also ψ Lösung von

$$\dot{z} = \frac{f(z) - z}{t}$$

Umgekehrt: Sei ψ Lösung von

$$\dot{z} = \frac{f(z) - z}{t}, \quad z(t_0) = \frac{x_0}{t_0}.$$

Setze nun

$$\varphi(t) := t\cdot\psi(t):$$

$$\dot{\varphi} = (t\psi)^{\cdot} = \psi + t\dot{\psi}$$

$$= \psi + (f(\psi) - \psi) = f(\psi)$$

$$= f\left(\frac{\varphi(t)}{t}\right) \quad \boxtimes$$

Problem: Setze $\dot{x} = h\left(\frac{x}{t}\right)$

$$z = \frac{x}{t}$$

Dann:

$$x = tz,$$

$$\dot{x} = z + t\dot{z}$$

Also:

$$z + t\dot{z} = h\left(\frac{x}{t}\right) = h(z)$$

Also:

$$\dot{z} = \frac{h(z) - z}{t}.$$

Beispiel:

$$\dot x = 1 + \frac{x}{t} + \frac{x^2}{t^2}, \qquad t \neq 0$$

$z = \frac{x}{t}$:

$$z + t\dot z = \dot x = f(t) = 1 + z + z^2$$

Also:

$$\dot z = \frac{1 + z^2}{t}, \qquad t \neq 0.$$

Also:

$$\frac{dz}{1+z^2} = \frac{dt}{t}$$

Also:

$$\arctan z = \log|t| + c$$

Also:

$$\frac{x}{t} = z = \tan(\log|t| + c)$$

Also

$$x(t) = t \cdot \tan(\log|t| + c).$$

2.

$$\dot{x} = \frac{x+\sqrt{t^2+x^2}}{t}, \qquad t>0$$

$$\frac{\lambda x + \sqrt{(\lambda t)^2+(\lambda x)^2}}{\lambda t} = \dots = \frac{x+\sqrt{t^2+x^2}}{t} \quad ✓$$

Fi: $z = \frac{x}{t}$:

$$z + t\dot{z} = \dot{x} = \frac{tz + \sqrt{t^2+t^2z^2}}{t}$$

$$= z + \sqrt{1+z^2}$$

:

$$\boxed{t\dot{z} = \sqrt{1+z^2}, \qquad t>0}$$

$$= \int \frac{dz}{\sqrt{1+z^2}} = \int \frac{dt}{t} = \log t + c$$

$$\log\left(z+\sqrt{1+z^2}\right)$$

Also:

$$z + \sqrt{1+z^2} = pt \quad , \quad p = e^c > 0$$

Damit

$$\sqrt{1+z^2} = pt - z \quad (\;)^2$$

$$1+z^2 = p^2t^2 - 2ptz + z^2$$

$$2pt\underline{z} = p^2t^2 - 1$$

Also:

$$\varphi(t) = \underbrace{\frac{p^2t^2-1}{2p}}_{} \quad , \quad p > 0$$

[illegible] für alle t

Tiefpunkt bei $-\frac{1}{2p}$

Brennpunkt bei 0.

Konfokale Parabeln.

$$\int \frac{dt}{\sqrt{1+t^2}}$$

Substitutie ?

$$t = \cancel{\tan t}$$

$$\left\{\begin{aligned} t &= \sinh t = \frac{e^t - e^{-t}}{2} \\ 1+t^2 &= \cosh^2 t \\ dt &= \cosh t \; dt \end{aligned}\right.$$

$$= \int dt = t = \operatorname{arcsinh} t + c$$

$$\sinh t = t$$

$$e^t - e^{-t} = 2t$$

$$\boxed{e^{2t} - 2t\,e^t - 1 = 0}$$