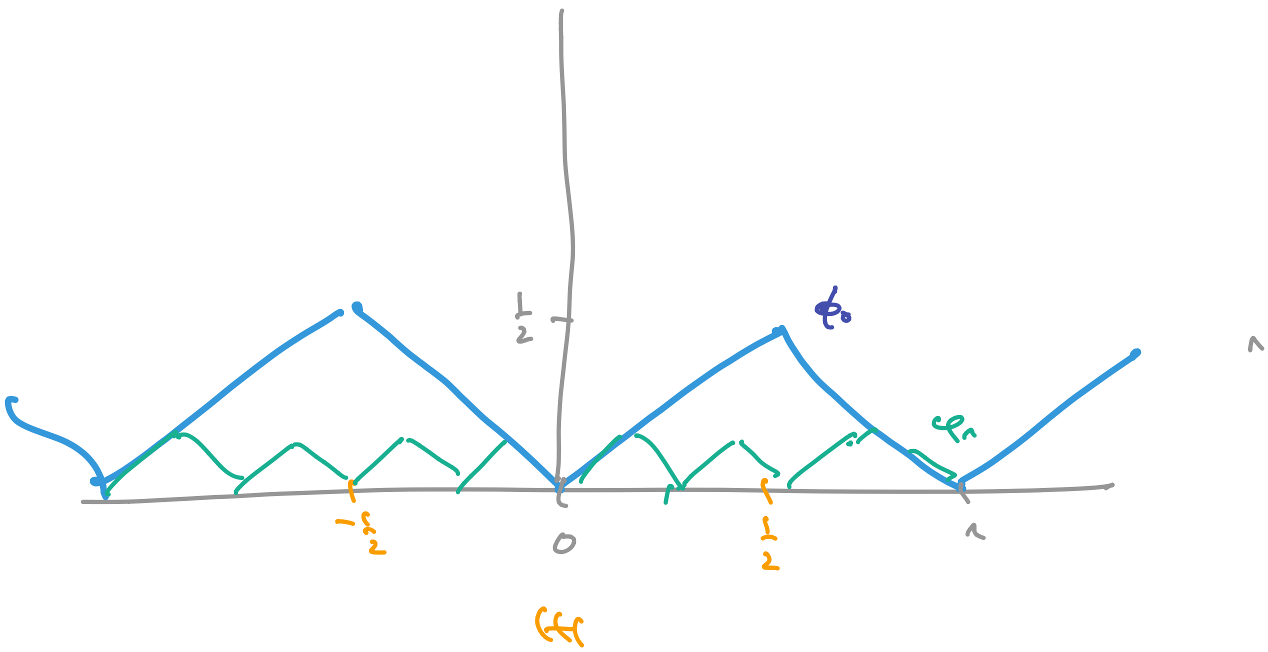
7. Vorlesung 10.5.2021



$$\varphi_0 : \mathbb{R} \to \mathbb{R} : \varphi_0(t) = |t - \underbrace{\lfloor t + \tfrac{1}{2} \rfloor}_{}|$$

periodisch mit 1.

$$\varphi_n(t) = \frac{1}{4^n}\,\varphi_0(4^n t) \quad , \quad n \geq 1$$

$$\boxed{\varphi : \mathbb{R} \to \mathbb{R} \, , \quad \varphi(t) = \sum_{n \geq 1} \varphi_n(t)}$$

Beweis: Stetigkeit:

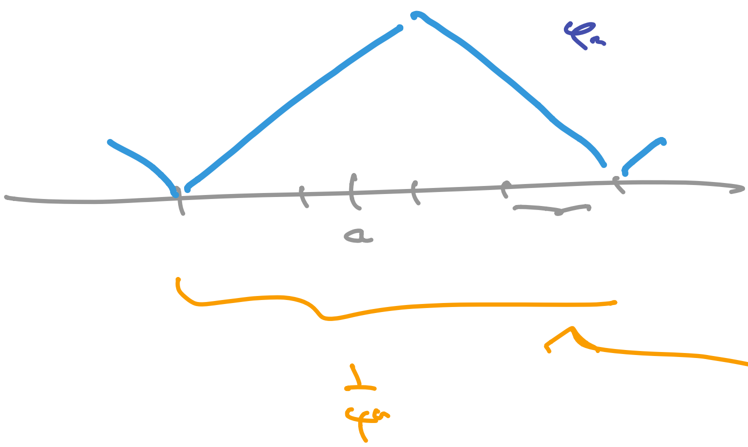
$$\left\| \sum_{k=n}^{m} \phi_k \right\|_R \le \sum_{k=n}^{m} \|\phi_k\|_R$$

$$\le \sum_{k=n}^{\infty} \frac{1}{2} \frac{1}{4^k} < \frac{1}{4^n}.$$

Nirgends Diff: $a \in \mathbb{R}$.

Diff-quot:

$$\frac{\phi_n(a+h_n) - \phi_n(a)}{h_n}$$



$h_n = \pm \frac{1}{4} \cdot \frac{1}{4^n}$:

a, $a+h_n$ in einem Teil von φ_n a

$$\left| \frac{\phi_n(a+h_n) - \phi_n(a)}{h_n} \right| = 1$$

Donc aussi

$$\left| \frac{\phi_k(a+h_n) - \phi_k(a)}{h_n} \right| = \begin{cases} 1, & 1 \le k \le n \\ 0, & k \ge n+1, \end{cases}$$

car ϕ_k période $\frac{1}{4^k}$.

Ainsi ici :

$$\frac{\phi(a+h_n) - \phi(a)}{h_n} = \underbrace{\sum_{k=1}^{n} \frac{\phi_k(a+h_n) - \phi_k(a)}{h_n}}_{\text{Série CF.}} \qquad \left(|\cdot| = 1\right)$$

Ne converge pas qd $n \to \infty$.

$\blacksquare$

Bem.:

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$$

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

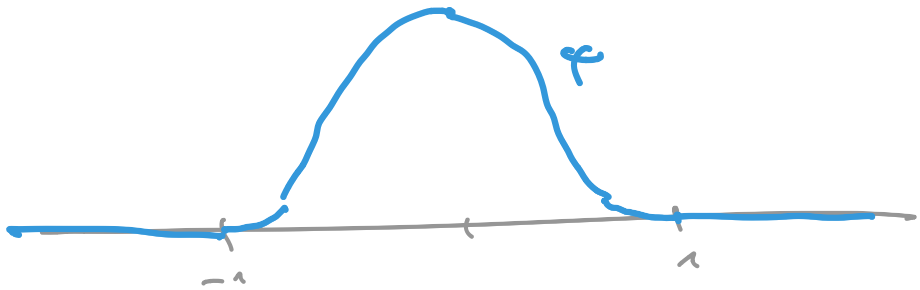
$$f^{(n)}(t) = p_n\left(\frac{1}{t}\right) e^{-\frac{1}{t^2}}, \qquad n \geq 1,$$

$$\operatorname{grad}(p_n) \leq 3n.$$

Da $t^m e^{-t} \longrightarrow 0$ für $t \to \infty$ für jedes $m \geq 0$.

$$\lim_{t \to 0} f^{(n)}(t) = 0, \qquad n \geq 1.$$

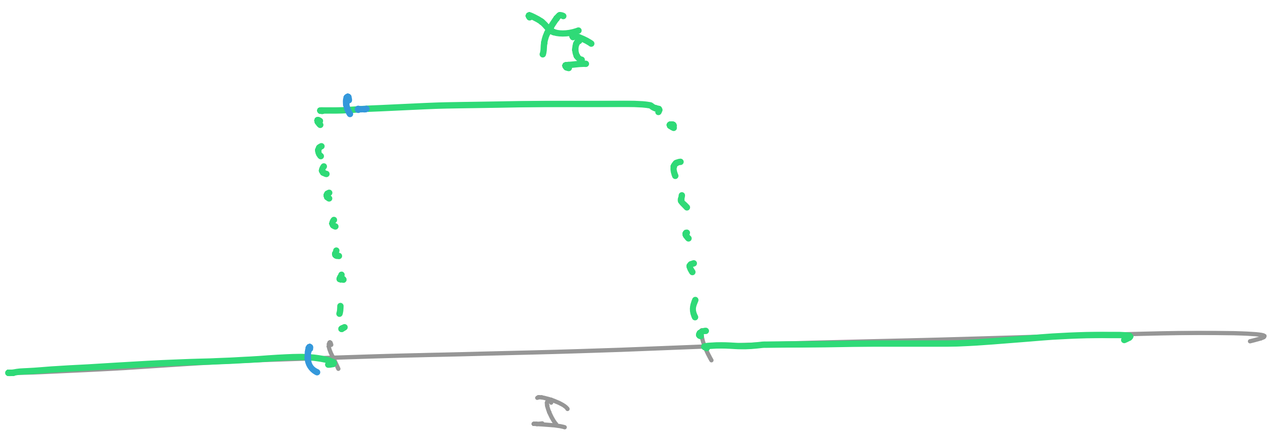
$$\varphi(t) = \begin{cases} e^{-\frac{t^2}{(1-t^2)}}, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$



C^∞-Funktion mit kompaktem Träger:

$$\operatorname{supp} \varphi = \overline{\{t \in \mathbb{R} : \varphi(t) \neq 0\}}$$

$$= [-1, 1]$$



Beweis: Sei $f \in C_b$, $g \in C_1$.

$r > 0$:

$$\int_0^r |f(x-t)\, g(t)|\, dt$$

$$\leq \|f\|_\infty \cdot \int_0^r |g(t)|\, dt$$

$$\leq \|f\|_\infty \cdot \|g\|_1, \qquad r > 0.$$

Also:

$$\lim_{r\to\infty} \int_0^r |\ldots| \quad \text{existiert.}$$

Analog für $\int_{-r}^0$ ∎

Beh.: $$f * g = g * f$$

Beweis mit Lemma 8: $t = x-s \iff s = x-t$

$$\int_{-r}^{r} f(x-t)\, g(t)\, dt = -\int_{x+r}^{x-r} f(s)\, g(x-s)\, ds$$

$$= \int_{x-r}^{x+r} f(s)\, g(x-s)\, ds$$

$r \to \infty$:

$$\int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds$$

Also: $$f * g = g * f.$$

$\varphi \in C_b(\mathbb{R})$ sei fest.

$$f \longmapsto f * \varphi$$

für jedes $f \in C_b(\mathbb{R})$.

$$T_\varphi : \quad f \longmapsto T_\varphi f = f * \varphi$$

Beweis: Linear. ✓

Beschränktheit:

$$|(T_\varphi f)(x)| \leq \int_{-\infty}^{\infty} |\underbrace{f(x-t)}\, \varphi(t)|\, dt$$

$$\leq \|f\|_\infty \underbrace{\int_{-\infty}^{\infty} |\varphi(t)|\, dt}$$

$$= \|f\|_\infty \, \|\varphi\|_1 \, , \quad x \in \mathbb{R}$$

Also:

$$\|T_\varphi f\|_\infty \leq \|f\|_\infty \cdot \|\varphi\|_1 .$$

Stetigkeit:

$$|T_\varphi f(x+h) - T_\varphi f(x)|$$

$$\le \int_{-\infty}^{\infty} |f(x+h-t) - f(x-t)|\,|\varphi(t)|\,dt$$

$$= \underbrace{\int_{-\infty}^{-r} + \int_{r}^{\infty}} + \underbrace{\int_{-r}^{r}} |\ldots|\,|\ldots|\,dt$$

Sei $\varepsilon > 0$.

$$\int_{-\infty}^{-r} + \int_{r}^{\infty} \ldots \le \|f\|_\infty \cdot \underbrace{\left(\int_{-\infty}^{-r} + \int_{r}^{\infty} |\varphi(t)|\,dt \right)}_{< \varepsilon \text{ für } r > R_0}$$

Da $\varphi \in C_s(\mathbb{R})$

Jetzt:

$$\int_{-r}^{r} |f(x+h-t) - f(x-t)|\,|\varphi(t)|\,dt$$

$f(\cdot + h) - f(\cdot)$

für $\underbrace{x-t}$: $|t| \le r$

in einem kompakten Intervall K

dort f gleichm. stetig:

Das heißt: für $\varepsilon > 0$ ex. ein δ:

$$|f(x+h-t) - f(x-t)| < \varepsilon$$

für $|h| < \delta$, $x-t \in K$.

$$\int_{-r}^{r} \underbrace{|f(x+h-t) - f(x-t)|}_{< \varepsilon} \, |\varphi(t)| \, dt$$

$$\leq \varepsilon \cdot \int_{-r}^{r} |\varphi(t)| \, dt$$

$$\leq \varepsilon \cdot \|\varphi\|_1 .$$

Also insgesamt:

$$|T_\varphi f(x+h) - T_\varphi f(x)| \leq \varepsilon \, \|f\|_\infty + \varepsilon \cdot \|\varphi\|_1$$

s. oben & hier

$$< \varepsilon \cdot c .$$

Also: $T_\varphi f$ unif. stetig.

∎

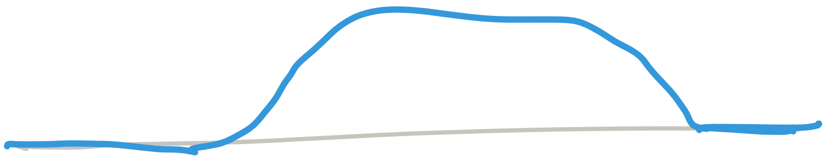
$$(T_\varphi f)(x) = \int_{-\infty}^{\infty} \varphi(x-t)\, f(t)\, dt$$

$$\begin{aligned}
(T_\varphi f)'(x) &= (\varphi * f)'(x) \\
&= \partial_x \int_{-\infty}^{\infty} \varphi(x-t)\, f(t)\, dt \\
&= \int_{-\infty}^{\infty} \partial_x \left(\varphi(x-t)\, f(t)\right) dt \\
&= \int_{-\infty}^{\infty} \partial_x \varphi(x-t)\, f(t)\, dt \\
&= f * \varphi' \qquad (\partial_x \varphi \in C_s) \\
&= T_{\varphi'} f .
\end{aligned}$$

" $\varphi \in C^n(\mathbb{R})$ und $\varphi^{(n)} \in C_b(\mathbb{R})$ "

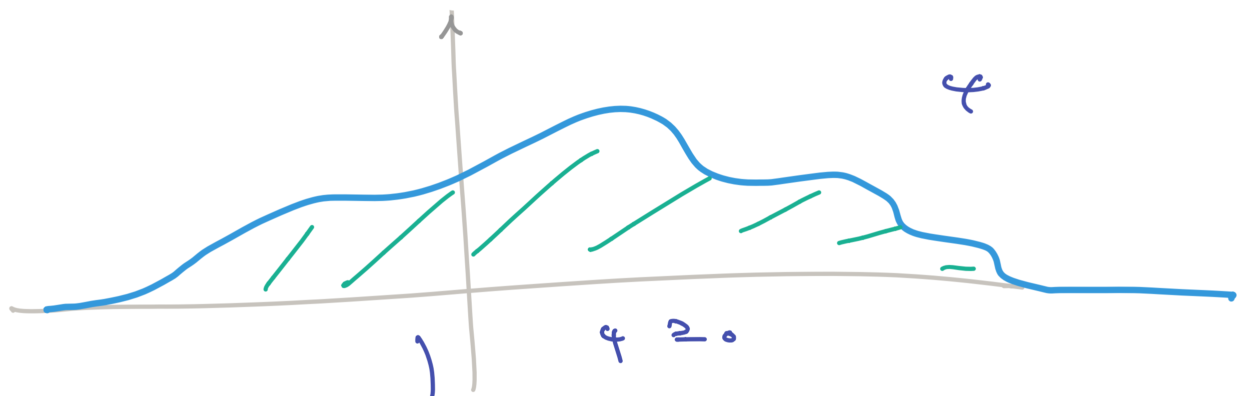
ist für jedes n erfüllt, wenn

$$\varphi \in C_0^\infty(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \operatorname{supp} f \text{ kompakt} \}.$$



(D-1) & (D-2) : Differenzierung

(D-3) :
$$\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \varphi_n(t)\,dt \longrightarrow 0 , \quad n \to \infty$$



φ

$$\left.\begin{array}{l} \varphi \geq 0 \\ \int \varphi(t) = 1 \\ \text{supp } \varphi \text{ kompakt} \end{array}\right\}$$

Defin: $\boxed{\varphi_n(t) = n\varphi(nt)}$

φ_n ist Diracfolge:

(D-1) $\varphi_n \geq 0$ ✓

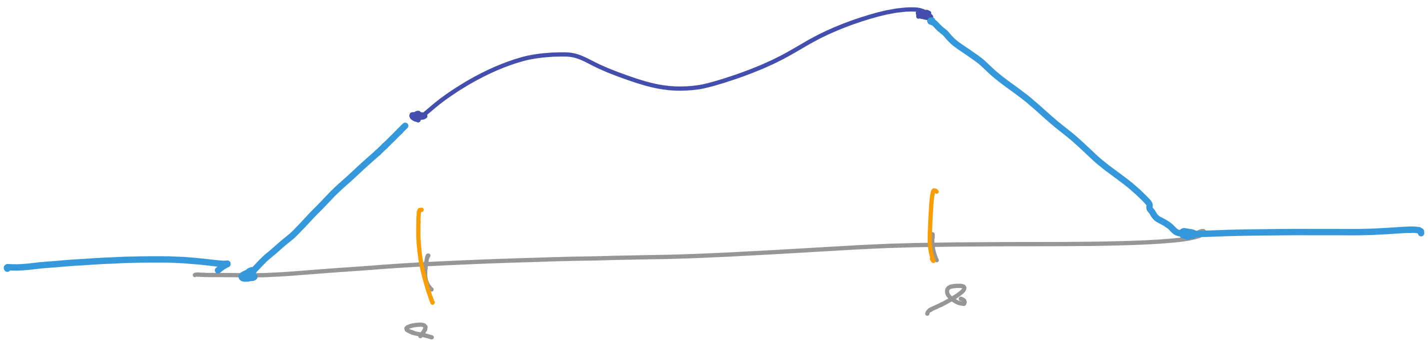
(D-2) $\int_{-\infty}^{\infty} \varphi_n(t) = \int_{-\infty}^{\infty} \underline{n}\,\varphi(nt)\,\underline{dt}$

$n\,dt = ds$

$nt = s$

$$= \int_{-\infty}^{\infty} \varphi(s)\,ds = 1$$

(D-3) $\int_{-\delta}^{\delta} \varphi_n(t)\,dt = \int_{-n\delta}^{n\delta} \varphi(t)\,dt \xrightarrow[n\to\infty]{} 1$.



Beweis: Sei $f_n = f * \varphi_n$.

$$f(x) = f(x) \cdot \int_{-\infty}^{\infty} \varphi_n(t)\,dt$$

$$= \int_{-\infty}^{\infty} f(x)\, \varphi_n(t)\,dt.$$

Dann:

$$f(x) - f_n(x) = \int_{-\infty}^{\infty} f(x)\varphi_n(t) - \int_{-\infty}^{\infty} f(x-t)\,\varphi_n(t)\,dt$$

$$= \int_{-\infty}^{\infty} \big(f(x) - f(x-t)\big)\, \varphi_n(t)\,dt.$$

Da $\varphi_n \geq 0$:

$$|f(x) - f_n(x)| \leq \int_{-\infty}^{\infty} |f(x) - f(x-t)| \cdot \varphi_n(t)\,dt.$$

$$= \int_{-\delta}^{\delta} + \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \xrightarrow{n} \;.$$

Sei $\varepsilon > 0$. Da f auf $\mathbb{R}$ glm. stetig:

ex. $\delta > 0$:

$$|f(x-t) - f(x)| < \varepsilon, \quad |t| < \delta, \quad x \in \mathbb{R}.$$

Damit:

$$\int_{-\delta}^{\delta} \underbrace{|f \dots|}_{<\varepsilon} \, \varphi_n(t)\, dt$$

$$< \varepsilon \cdot \int_{-\delta}^{\delta} \varphi_n(t)\, dt < \varepsilon .$$

Für den Rest:

$$\leq 2 \|f\|_\infty \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \varphi_n(t)\, dt \right)$$

$$= 2 \|f\|_\infty \Big(\underbrace{1 - \int_{-\delta}^{\delta} \varphi_n(t)\, dt}_{(D-3):\ \xrightarrow[n\to\infty]{} 0} \Big)$$

$$< 2 \|f\|_\infty \cdot \varepsilon , \qquad n \geq N .$$

Damit gezeigt:

$$|f_n(x) - f(x)| \longrightarrow 0 \quad \text{glm. für } x .$$