9. Vorlesung 17.5.2021

$C^r(I, E)$: r-mal stetig diffbare K.

$0 \le r \le \infty$

$$\left.\begin{array}{l}\gamma : I \longrightarrow E \\ \varphi : J \longrightarrow I\end{array}\right\} \quad \underbrace{\gamma \circ \varphi} : J \longrightarrow E$$

$$(\gamma \circ \varphi)^{\cdot} = \underbrace{(\dot\gamma \circ \varphi)}_{\in E} \cdot \underbrace{\dot\varphi}_{\text{Skalar}}$$

$\gamma : I \longrightarrow \mathbb{R}^m$,

$\gamma = (\gamma_1, \dots, \gamma_m)$

γ stetig $\Longleftrightarrow$ $\gamma_1, \dots, \gamma_m$ stetig.

Dasselbe für Diff. :

$\dot\gamma(a) = (\dot\gamma_1, \dots, \dot\gamma_m(a))$.

Bsp: 1.

$$\gamma : [0, 2\pi) \longrightarrow \mathbb{R}^2$$
$$\gamma(t) = (\cos t, \sin t).$$

$$\dot\gamma(t) = (-\sin t, \cos t)$$

$$\|\dot\gamma(t)\|_2 = 1.$$

2. $f : I \longrightarrow \mathbb{R}$

Graph als Kurve:

$$\gamma : I \longrightarrow \mathbb{R}^2, \quad \gamma(t) = (t, f(t))$$

γ ist C^k genau ~~dann~~, wenn f C^k,

$$\dot\gamma(t) = (1, f'(t))$$

$$\|\dot\gamma(t)\|_2 = \sqrt{1 + (f'(t))^2}.$$

Lipschitzeigenschaft:

$$\| J_I(\varphi) \| \le \sum_{k=1}^{n} \|c_k\| \, (t_k - t_{k-1})$$

$$\le \sup_I \|\varphi\| \cdot (b-a)$$

$$\gamma : I \longrightarrow E$$

$$\varphi_n \rightrightarrows \gamma$$

$$\int_I \gamma = J_I(\gamma) := \lim_{n\to\infty} J_I(\varphi_n) .$$

Dreiecksungleichung:

$$\Big\| \underbrace{\int_I \gamma}_{E} \Big\|_E \le \int_I \underbrace{\|\gamma\|_E}_{\mathbb{R}}$$

$$\gamma : I \to \mathbb{R}^m$$

$$\gamma = (\gamma_1, \dots, \gamma_m)$$

$$\int_I \gamma = \int_I (\gamma_1, \dots, \gamma_m)$$

$$= \left(\int_I \gamma_1, \dots, \int_I \gamma_m \right).$$

Bsp.:

$$\gamma : \mathbb{R} \to \mathbb{R}^3$$

$$\gamma(t) = (2t, 3t^2, 4t^3)$$

$$\Gamma(t) = \int_0^t \gamma$$

$$= \int_0^t (2s, 3s^2, 4s^3)$$

$$= \left(\int_0^t 2s, \int_0^t 3s^2, \int_0^t 4s^3 \right) = (t^2, t^3, t^4).$$

Beweis: $\gamma : I \to \mathbb{R}^n$ stetig,

dann auch integrierbar.

$$\Phi(t) = \int_c^t \gamma \quad \text{wohldefiniert.}$$

$$\Phi(t+h) = \int_c^t \gamma + \int_t^{t+h} \gamma$$

$$= \Phi(t) + \int_t^{t+h} \gamma$$

Also:

$$\frac{1}{h}\left(\Phi(t+h) - \Phi(t)\right) = \frac{1}{h}\int_t^{t+h} \gamma .$$

$$\xrightarrow[h \to 0]{} \gamma(t).$$

Also: Φ ist diffbar,

$$\Phi'(t) = \gamma(t). \qquad \square$$

Beweis: [illegible] [illegible] [illegible]:

[illegible] [illegible] [illegible]:

$$F(b) - F(a) = \int_a^b f$$

$$= \int_c^b f - \int_c^a f$$

$$= \Gamma(b) - \Gamma(a)$$

Für [illegible] [illegible] Stammf.

Bem.: $\gamma \in C^1(I, \mathbb{R}^n)$,

$a < b$ in I.

$$\gamma(b) - \gamma(a) = \int_a^b \dot\gamma$$

Also:

$$\|\gamma(b) - \gamma(a)\| = \left\| \int_a^b \dot\gamma \right\|$$

$$\leq \int_a^b \|\dot\gamma\|$$

stetig in t

$$\leq \max_{a \leq t \leq b} \|\dot\gamma(t)\| \cdot (b-a)$$

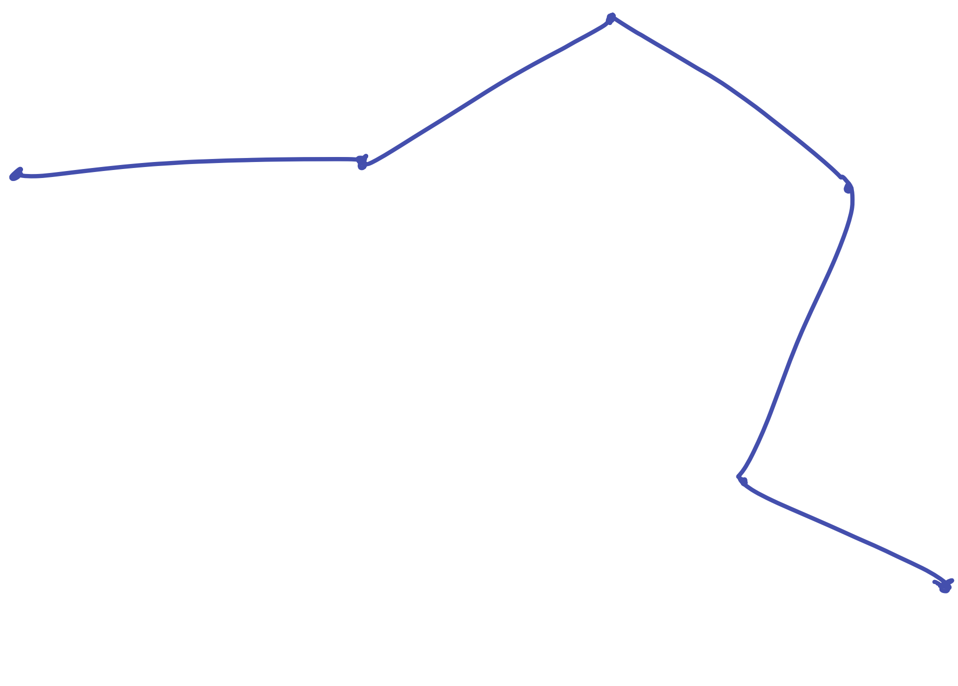
$$= \underbrace{\|\dot\gamma\|_{[a,b]}}_{< \infty} \cdot (b-a).$$

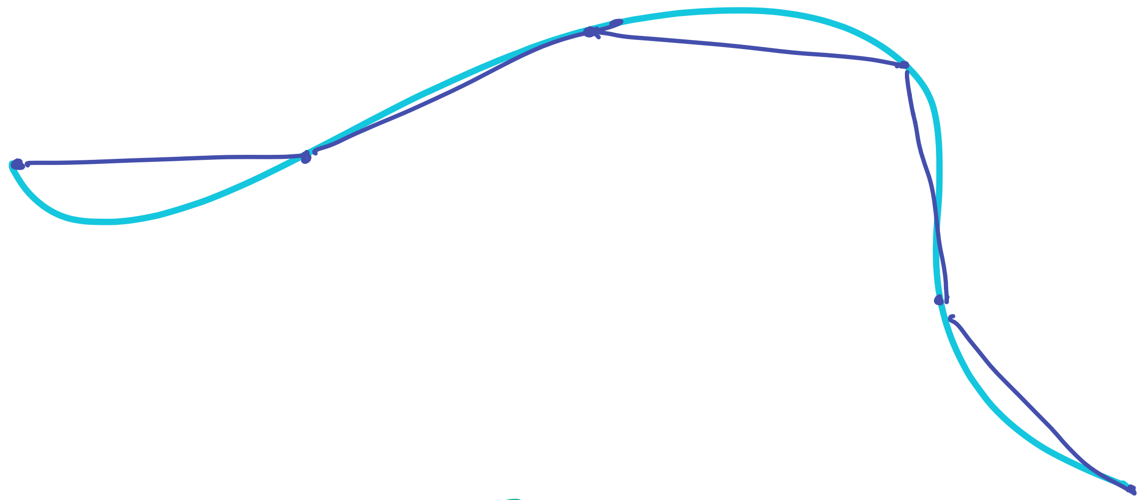
$$\underbrace{f(v) - f(u) = f'(\xi)(v-u)}_{\text{Mittelwertsatz}}$$

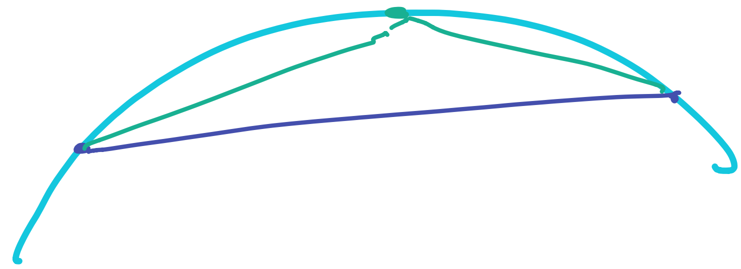
$$| \ldots | \le |\ldots| (v-u)$$

Mittelwertsatz für Kurven gilt so nicht:

$$\cancel{\gamma(v) - \gamma(u) = \gamma'(\tau)(v-u)}$$





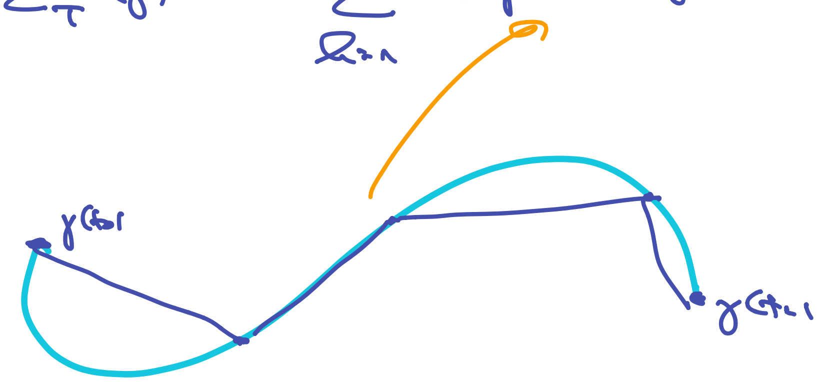


$I = [a,b]$

$\gamma : I \longrightarrow E$

$T = (t_0, \dots, t_n)$ Tagging on I :

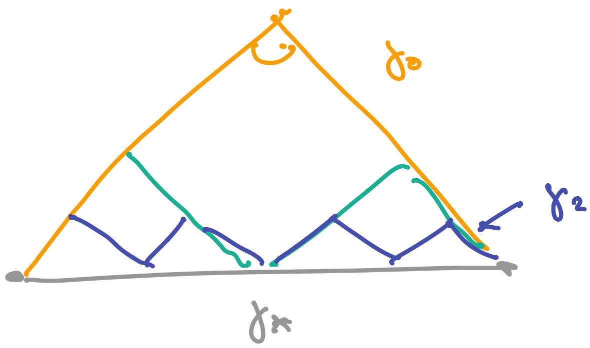
$$\Sigma_T(\gamma) = \sum_{k=1}^{n} \| \gamma(t_k) - \gamma(t_{k-1}) \|_E$$



$$L_I : C(I, E) \longrightarrow [0, \infty]$$

$$\gamma \longmapsto L_I(\gamma)$$

stetig ?   Nein !   Gegenbsp.



$$L_I(\gamma_*) = 1 .$$

$$\| \gamma_n - \gamma_* \| \longrightarrow 0$$

Aber: $\quad L(\gamma_0) = \sqrt{2} = L(\gamma_n) \; , \; n \geq 1$

$$L(\gamma_n) \not\longrightarrow L(\gamma_*) .$$

Sei $f : I \to E$ M-Lipschitz:

$$\|f(x) - f(y)\|_E \le M \cdot |x-y|.$$

Teilung T von I:

$$\Sigma_T(f) = \sum_{k=1}^{n} \|f(t_k) - f(t_{k-1})\|_E$$

$$\le M \cdot \sum_{k=1}^{n} (t_k - t_{k-1})$$

$$\le \underbrace{M \cdot |I|.}_{\text{unabhängig von } T.}$$

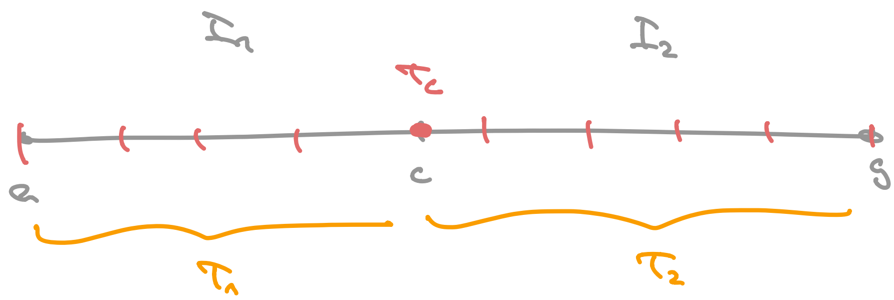
Also:

$$\sup_T \Sigma_T(f) \le M \cdot |I| \ ,$$

f C^1-Kurve:

$$\|f(x) - f(y)\| \le \underbrace{\sup_I \|f'\|}\ |x-y|$$

Beweis:



$I_1 = [a, c], \quad I_2 = [c, b].$

Z.z.:

$$L_I = L_{I_1} + L_{I_2}$$

Sei $T = \{t_0, \dots, t_n\}$, $\quad T_c = T \cup \{c\}$

$$\Sigma_T \leq \Sigma_{T_c} = \Sigma_{T_1} + \Sigma_{T_2}$$
$$\leq L_{I_1} + L_{I_2}$$

Also:

$$L_I \leq L_{I_1} + L_{I_2}.$$

z.z. $\qquad c_I \geq c_{I_1} + c_{I_2}$.

Zunächst: $\qquad c_{I_1} + c_{I_2} < \infty$.

$\varepsilon > 0$. $\qquad \Sigma_1 \quad \exists \quad T_1, T_2$:

$$\Sigma_{T_i} \geq c_{I_i} - \frac{\varepsilon}{2}, \qquad i = 1, 2.$$

Für $T = T_1 \cup T_2$:

$$\Sigma_T = \Sigma_{T_1} + \Sigma_{T_2} \geq c_{I_1} + c_{I_2} - \varepsilon.$$

$\sup_T$:

$$c_I \geq c_{I_1} + c_{I_2} - \varepsilon.$$

Für jedes $\varepsilon > 0$: $\qquad \Rightarrow$:

$$c_I \geq c_{I_1} + c_{I_2}. \qquad \square$$

Beweis: Für $a<b$ in I:

$$\|\gamma(b)-\gamma(a)\|_E = \left\| \int_a^b \dot\gamma \right\| \le \int_a^b \|\dot\gamma\| .$$

Für $J \subset I$, $T \subset J$:

$$\Sigma_T(\gamma) = \sum_{k=1}^{n} \|\gamma(t_k)-\gamma(t_{k-1})\|_E \le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\dot\gamma\|_E = \int_J \|\dot\gamma\|$$

$$\Longrightarrow \quad \boxed{L_J(\gamma) \le \int_J \|\dot\gamma\| .}$$

Sei $t \in [a,b)$, $h>0$: $t+h \le b$.

Dann gilt:

$$\|\gamma(t+h)-\gamma(t)\|_E \le L_{[t,t+h]}(\gamma) \le \int_t^{t+h} \|\dot\gamma\|_E$$

$$L_{[a,t+h]}(\gamma) - L_{[a,t]}(\gamma)$$

$$= \lambda(t+h) - \lambda(t).$$

Dividiere durch $h>0$:

$$\left\|\frac{\gamma(t+h)-\gamma(t)}{h}\right\|_E \le \frac{\lambda(t+h)-\lambda(t)}{h} \le \frac{1}{h}\cdot\int_t^{t+h} \|\dot\gamma\|_E$$

Da $\gamma \in C^1$: für $h \to 0$:

$$\|\dot\gamma(t)\|_E = \dot\lambda(t) = \|\dot\gamma(t)\|_E$$

Beweis: Siehe Bew. von Schritt Lemma:

$$L_I(\gamma) = \lambda(b)$$

$$= \lambda(b) - \underbrace{\lambda(a)}_{=0}$$

$$= \int_a^b \lambda'(t)\, dt$$

$$= \int_I \| \gamma'(t) \|_E$$